REMARKS ON CURVATURE BLOW-UP AND FUNDAMENTAL TONES ON MULTI-BUBBLES

These remarks address some of the pitfalls which, while not present in the single-bubble setting, do arise in the multi-bubble variational setting in regards to curvature potentially blowing-up near quadruple points, as well as working with fundamental tones (a.k.a. ground states) of the Jacobi operator. Below is a non-comprehensive list of some of the main issues in these directions which we have found in Heilman's paper on the topic, "The Structure of Gaussian Minimal Bubbles", up until Section 5 (unfortunately, there are additional issues afterwards as well). The list culminates with (6) – the most serious error in the argument, which we are afraid seems detrimental.

We will use notation used in that paper despite having issues with some of it (like $\partial\partial\Sigma_{ij}$). Recall that Σ_{ij} denotes the interface between the cells Ω_i and Ω_j, and that C denotes the locus of meeting points of three cells.

(1) Differentiating inside the integral in the second-variation Lemma 3.7 is not justified at all, and as one knows, exchanging limits with integration is definitely not automatic and sometimes false. We do not see how the usual convergence theorems (monotone, dominated) would apply here; at the very least, this would require the a-priori integrability of the curvature $||A||^2$ on Σ_{ij} (Heilman deduces this integrability in Lemma 5.9, but that would be a circular argument). Even assuming this, some care should be taken and an appropriate justification should be supplied, as this seems to be a genuine and not merely technical issue. Previous works from the single-bubble ($m = 2$) or double-bubble ($m = 3$) settings do not have this issue since the boundary of a minimizer is typically smooth outside a singular set of small Hausdorff dimension, and this bad set can be safely truncated without influencing the second variation; however, this is not the case in the triple-bubble (and more generally $m \geq 4$) setting in Heilman's paper, where already $\partial\partial\Sigma_{ij}$ is only assumed to be $C^{1,\alpha}$ smooth according to Assumption 2.4, and so curvature might be blowing up near this set. According to Remark 1.4 in the Colombo–Edelen–Spolaor paper which is used to corroborate Assumption 2.4, these authors suspect that the $C^{1,\alpha}$ regularity of $\partial\partial\Sigma_{ij}$ is sharp, in contrast to that of $\partial\Sigma_{ij}$ which may be upgraded to C^∞, and so this is not merely a technical issue. This implicit usage of the a-priori integrability of curvature appears in numerous additional places (see e.g. below).

(2) Definition 4.2 of the quadratic form $Q(f, g)$ and the integration-by-parts in Lemma 4.3: Q may be undefined even when the compactly supported f and g vanish on $\partial\partial\Sigma_{ij}$ (since their supports can still intersect this set). To ensure

that Q is finite, one needs to *a-priori* assume the L^2 integrability of curvature ($\|A\|^2$) on Σ_{ij} (as in (1) above) **as well as** the L^1 integrability of curvature (q_{ij}) on $\partial^*\Sigma_{ij}$. The finiteness of $Q(f,g)$ for $f,g \in C_0^\infty(\cup\Sigma_{ij})$ is used throughout the paper, for instance in the proofs of Lemmas 5.6 and 5.4, where a cancellation occurs (which is perfectly fine if the terms being canceled are finite, but if they are infinite, the resulting identity is invalid, i.e. $1 + \int_0^1 \frac{dx}{x} = 2 + \int_0^1 \frac{dx}{x}$ does not imply that $1 = 2$; nor can one claim that $a_i \int_0^1 \frac{dx}{x}$ sums to 0 if a_i sum to zero).

(3) Lemma 3.12 is quoted from [HMRR02] - but these authors only treated the double-bubble case $m = 3$, and so did not have all of the additional intricate structure in meeting points of 4 or more partition elements given by Assumption 2.4, and in particular the curvature q_{ij} was (locally) bounded away from the singular set, ensuring that the integral appearing in the conclusion of Lemma 3.12 converges. We don't see how this applies to general $m \geq 4$ without additional justification, and the previous comments in (1) and (2) above apply as well.

(4) Lemma 3.9 (extension of functions to vector fields) - the statement and proof are false for functions $f_{ij} \in C_0^\infty(\Sigma_{ij})$ satisfying the compatibility condition (26) on C, due to the fact that the sets $\partial\partial\Sigma_{ij}$ are only assumed to be $C^{1,\alpha}$ smooth in Assumption 2.4, and hence N_{ij} is only $C^{0,\alpha}$ smooth near these sets. This means that: A. the extended fields $\tilde{N}_{ij}$ cannot be guaranteed to be smooth on $\mathbb{R}^{n+1}$; B. the field Z which is defined pointwise by $\langle Z, N_{ij}\rangle = f_{ij}$ on C cannot be guaranteed to be smooth on $\cup\Sigma_{ij}$ all the way up to $\partial\partial\Sigma_{ij}$. Both $\tilde{N}_{ij}$ and Z are used in the construction of $\tilde{\Psi}$ and hence Ψ, and so the resulting field X is not smooth in a neighborhood of $\partial\partial\Sigma_{ij}$, let alone Lipschitz there (to generate a well-defined flow).

This extension is of course crucially used throughout the paper whenever invoking minimality, or more generally stability, i.e. the non-negativity of $Q(f,f)$ whenever $f = f_{ij}$ integrates to 0 on Σ.

Actually, this raises the issue of what is meant by a function $f_{ij} \in C_0^\infty(\Sigma_{ij})$ if $\partial\partial\Sigma_{ij}$ is only assumed to be $C^{1,\alpha}$ smooth, with the problematic point being the C^∞ smoothness of f_{ij} on the non-smooth manifold-with-boundary-with-boundary (i.e. with two-codimensional corners) Σ_{ij} – we have tried several definitions (such as restricting a C^∞ function defined on $\mathbb{R}^{n+1}$ to Σ_{ij}) but none of them are simultaneously compatible with Lemma 3.9 and e.g. the construction of $f_{ij} = \langle v, N_{ij}\rangle$ in Remark 5.1 or F_p in Lemma 5.4.

(5) In various additional places a statement is made for functions of compact support or which vanish on an appropriate problematic set, but then it is applied to functions which do not satisfy this important assumption. It seems that a naive attempt to truncate near these points will fail since $\partial\partial\Sigma_{ij}$ is the meeting locus of four partition elements, and has positive $(n-2)$-dimensional Hausdorff measure, and so will influence the second variation.

For example, in Lemmas 5.6 and 5.7 it is assumed that $\Phi \in C_0^\infty(\Sigma)$, but for the proof of Lemma 5.7, Heilman invokes the truncation Lemma 4.4 where it is (correctly) assumed that η (which incidentally should be defined on Σ, not merely $\cup\partial^*\Omega_i$ as stated) should be supported in $M_n \cup M_{n-1} \cup M_{n-2}$; consequently, it is not enough to only assume that $\Phi = 0$ on $M_{n-2} \cup M_{n-3}$, since the latter set has positive $(n-2)$-dimensional Hausdorff measure and the truncation argument does not apply.

(6) We believe it is possible to correct or circumvent the above problems by using some of the technical results we develop in our paper "The Gaussian Double-Bubble and Multi-Bubble Conjectures". However, the most serious problem, which we are not sure how to resolve, is in the key construction from Lemma 5.3 of the fundamental tone F, which is supposed to be a minimizer of the quadratic form Q. The construction of an eigenfunction F, satisfying $LF = \delta(\Sigma)F$ under the assumption that $\delta(\Sigma) < \infty$, is plausible; F is obtained as the limit of Dirichlet eigenfunctions F_k on the approximating manifolds Σ_k. However, why does it follow that this F is a minimizer of the form Q (which involves a *boundary* term)? Regardless of whether one uses Definition 4.2 or Lemma 4.3 as the definition of Q, there is no reason for the boundary term in (29) to vanish (either directly or after integration-by-parts, respectively), since the required boundary conditions for this to happen are incompatible with the Dirichlet boundary conditions satisfied by the limiting function F. The required boundary conditions for $F = \{f_{ij}\}$ on C, besides $f_{ij} + f_{jk} + f_{ki} = 0$, are:

$$\nabla_{\nu_{ij}} f_{ij} + q_{ij} f_{ij} = \nabla_{\nu_{jk}} f_{jk} + q_{jk} f_{jk} = \nabla_{\nu_{ki}} f_{ki} + q_{ki} f_{ki}, \tag{0.1}$$

which are stated as the conclusion of Lemma 5.3 and derived from the presupposition that F minimizes Q, which is a circular argument.

Actually, it is not even clear where the Dirichlet boundary conditions on F_k are imposed, since this depends on what is meant by "connected components of $\Sigma = \cup_{i<j}\Sigma_{ij}$". Either: A. every connected component is a subset of some Σ_{ij} (this is the literal interpretation, which is also consistent with the proof that $\delta \geq 1$ in Lemma 5.4, since the functions F_p there do not have constant sign across different Σ_{ij}'s), in which case the resulting F will necessarily satisfy Dirichlet boundary conditions on C as the limit of such F_k's, and would violate (0.1) even for the actual minimizing model simplicial clusters – see below; Or B. The connected components of Σ_k are allowed to span several Σ_{ij}'s with boundary condition on C given by the constraint $f_{ij} + f_{jk} + f_{ki} = 0$ of $\mathcal{F}$, but then: (i) it is not possible to guarantee that F_k does not change sign on each connected component of Σ_k, since the usual argument (for establishing the positivity of the first Dirichlet eigenfunction) of flipping the sign on each nodal domain would potentially violate the above boundary constraint; (ii) this would invalidate the proof of Lemma 5.4 as explained above; (iii) the PDE $Lf = \delta f$ would be under-determined, since a necessary condition for its well-posedness

is to have three linearly independent constraints relating f and ∇f on C.

A potentially insightful example, which demonstrates that an approximation procedure with Dirichlet boundary conditions on C does not produce a fundamental tone, is given by a standard Gaussian double-bubble in $\mathbb{R}^2$. In that case C consists of a single point. The approximation procedure (possibility A above) will result in a function $F = \{f_{ij}\}$ which vanishes on C. Assume in the contrapositive that F is a fundamental tone satisfying $LF = \delta F$ with $\delta = \delta(\Sigma)$. Since the interfaces are flat, the Jacobi operator L boils down to $LF = \Delta_{\Sigma,\gamma} F + F$. Integrating by parts, thanks to F vanishing on C, we have:

$$\delta \int_\Sigma F^2 d\gamma = \int_\Sigma FLF d\gamma = \int_\Sigma (F^2 - |\nabla F|^2) d\gamma,$$

and we deduce that $\delta \leq 1$. But as we also know (by testing constant functions as in Lemma 5.4), $\delta(\Sigma) \geq 1$ for a fundamental tone, so $\delta = 1$ and necessarily $\nabla F = 0$ on Σ. Hence F must be constant on each interface. But as it vanishes on C, it must be zero throughout, in contradiction to the condition $F_k(x) = 1$ which was imposed throughout the approximation at a fixed point x. This means that the constructed F is not a fundamental tone. In particular, this also means that this procedure cannot produce an F which satisfies (0.1) on C, since if it did, this F would indeed be a minimizer of Q and hence a fundamental tone. The same argument applies to any standard Gaussian m-bubble in $\mathbb{R}^{n+1}$ for $m \leq n+2$.

Regardless, since a minimizer of Q *automatically* satisfies the desired boundary conditions (0.1) on C, it would be quite miraculous if the constructed F would automatically satisfy these boundary conditions, even though they were never enforced along the approximation by the Dirichlet eigenfunctions F_k. One way to guarantee that the constructed eigenfunction F is indeed a minimizer of Q would be to enforce (0.1) along the approximation and establish the continuity of Q (or at the very least weak semi-continuity) on the appropriate Sobolev space, or to establish that Q is closed, but this seems like a genuine technical and possibly conceptual challenge if the curvature is unbounded in a neighborhood of $\partial\partial\Sigma_{ij}$, even assuming one shows it is integrable. Moreover, even if one manages to overcome this challenge, the sign of F on each interface Σ_{ij} would not be guaranteed to remain constant in this approach, and this seems like a detrimental blow to the entire argument.

Finally, it should be noted that Heilman's main supplementary Assumption 1.6 does not even hold on the model case (i.e. for the actual minimizers) whenever $n+1 \geq m$, since in that case the singular set C (which has Hausdorff dimension $n-1$) is unbounded. Fortunately, we think this can be resolved by appropriately modifying the assumption.