ISOPERIMETRY, SOBOLEV AND CONCENTRATION INEQUALITIES THROUGH THE LENS OF CONVEXITY

(note that the statement of one exercise may be used for the proof of another).

(1) Prove that $det^{1/n}$ is concave on the class of $n \times n$ positive semi-definite matrices:

$$A, B \geq 0 \quad \Rightarrow \quad det(A+B)^{1/n} \geq det(A)^{1/n} + det(B)^{1/n} \ .$$

(2) Recall that the collection $\mathcal{K}$ of all compact subsets of a complete separable metric space (Ω, d), equipped with the Hausdorff distance $\mathcal{H}(K_1, K_2) = \inf \{\varepsilon > 0 \ ; \ K_1 \subset (K_2)_\varepsilon \text{ and } K_2 \subset (K_1)_\varepsilon\}$, is a complete metric space. Also, recall that a Borel measure μ on (Ω, d) is called inner regular if for any Borel set A:

$$\mu(A) = \sup \{\mu(K) \ ; \ A \supset K \text{ compact}\} \, ,$$

and outer regular if:

$$\mu(A) = \inf \{\mu(G) \ ; \ A \subset G \text{ open}\} \, .$$

Recall that the Lebesgue measure Vol on Euclidean space is both inner and outer regular.

(a) Show that any outer regular measure μ is upper semi-continuous on $(\mathcal{K}, \mathcal{H})$:

$$K_i \to_{\mathcal{H}} K \quad \Rightarrow \quad \mu(K) \geq \limsup_{i \to \infty} \mu(K_i).$$

(b) We have shown in class the Brunn–Minkowski (BM) inequality for sets A, B which are unions of axis-aligned boxes. Deduce the BM inequality for all compact sets $A, B \subset \mathbb{R}^n$.

(c) Deduce the BM inequality for all Borell sets $A, B \subset \mathbb{R}^n$.

(3) Recall that the Prékopa–Leindler (PL) inequality states that if $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ denote three measurable functions so that for some $\lambda \in (0,1)$:

$$h(\lambda x + (1-\lambda) y) \geq f(x)^\lambda g(y)^{1-\lambda} \quad \forall x, y \in \mathbb{R}^n \ .$$

Then:

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f \right)^\lambda \left(\int_{\mathbb{R}^n} g \right)^{1-\lambda} \ .$$

We've seen in class that the n-D PL inequality implies the n-D BM inequality. The goal of this exercise is to show the reverse implication. We've already seen that in class that the 1-D BM inequality implies the 1-D PL inequality. Generalize this to arbitrary dimension n using two different methods:

(a) Method 1. Apply the BM inequality in $\mathbb{R}^n$ to the level-sets of h, f, g, and conclude using the 1-D PL inequality (you will need a slightly different variant of the usual 1-D PL, which is equivalent to it).

(b) Method 2. First prove, using the BM inequality in $\mathbb{R}^{n+k}$, that if:

$$h(\lambda x + (1-\lambda)y)^{\frac{1}{k}} \geq \lambda f(x)^{\frac{1}{k}} + (1-\lambda) g(y)^{\frac{1}{k}} \quad \forall x, y \in \mathbb{R}^n$$

for some natural number k, then:

$$\left(\int_{\mathbb{R}^n} h\right)^{\frac{1}{n+k}} \geq \lambda \left(\int_{\mathbb{R}^n} f\right)^{\frac{1}{n+k}} + (1-\lambda)\left(\int_{\mathbb{R}^n} g\right)^{\frac{1}{n+k}} .$$

(do not use the statement of other exercises for the proof). Conclude the n-D PL inequality by taking $k \to \infty$ and using that for $a, b > 0$:

$$\lim_{\varepsilon \to 0} (\lambda a^{\varepsilon} + (1-\lambda) b^{\varepsilon})^{1/\varepsilon} \to a^{\lambda} b^{1-\lambda} .$$

(4) C. Borell generalized the BM/PL inequalities as follows: let $\Psi_\lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$ denote a continuous, increasing in each variable, and 1-homogeneous function (i.e. $\Psi_\lambda(tx, ty) = t\Psi_\lambda(x, y) \quad \forall t > 0$). Let $f_0, f_1, f_\lambda \geq 0$ denote continuous integrable functions on $\mathbb{R}^n$, so that $\int f_0, \int f_1 > 0$. Then the following statements are equivalent:
(a) For all compact $A_0, A_1 \subset \mathbb{R}^n$:

$$\int_{(1-\lambda)A_0 + \lambda A_1} f_\lambda \geq \Psi_\lambda\left(\int_{A_0} f_0, \int_{A_1} f_1\right).$$

(b) For all $x_0, x_1 \in \mathbb{R}^n$, for all $a^0, a^1 \in \mathbb{R}_+^n$ (the positive orthant):

$$f_\lambda((1-\lambda)x_0 + \lambda x_1)\Pi_{i=1}^n((1-\lambda)a_i^0 + \lambda a_i^1) \geq \Psi_\lambda(f_0(x_0)\Pi_{i=1}^n a_i^0, f_1(x_1)\Pi_{i=1}^n a_i^1).$$

Note that the function $\Psi_\lambda(x, y) = ((1-\lambda)x^{1/n} + \lambda y^{1/n})^n$ corresponds to the BM inequality, whereas the function $\Psi_\lambda(x, y) = x^{1-\lambda} y^{\lambda}$ corresponds to the PL inequality.

The proof that (a) implies (b) is immediate by testing (a) on small axis-aligned boxes. Assuming that (b) implies (a) indeed holds in dimension 1, show that (b) implies (a) holds in any dimension n, using two different methods:
(a) Induction on the dimension n.
(b) Proving (a) for sets A_0, A_1 which are unions of k_0, k_1 small axis-aligned boxes (respectively) having disjoint interiors, using induction on $k_0 + k_1$.

(5) Using Borell's theorem in the previous exercise, find the optimal function $\Phi_{N,n,\lambda} : \mathbb{R}_+^2 \to \mathbb{R}_+$, so that for all f_0, f_1, f_λ as above satisfying:

$$f_\lambda((1-\lambda)x_0 + \lambda x_1) \geq \Phi_{N,n,\lambda}(f_0(x_0), f_1(x_1)) \quad \forall x_0, x_1 \in \mathbb{R}^n,$$

it follows that:

$$\int f_\lambda \geq \Psi_{N,\lambda}\left(\int f_0, \int f_1\right),$$

where $\Psi_{N,\lambda} = ((1-\lambda)x^{1/N} + \lambda y^{1/N})^N$ for some fixed $N \geq n$. The resulting inequality is called the Borell / Brascamp–Lieb inequality (compare with exercise 3 (b)).

(6) We've seen in class that if $f : \mathbb{R}^n \to \mathbb{R}_+$ is a log-concave function:
$$\forall \lambda \in [0,1] \;\; \forall x, y \in \mathbb{R}^n \quad f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda} \; ,$$
then $\mu = f(x)dx$ is a log-concave measure:
$$\forall A, B \subset \mathbb{R}^n \quad \mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda} \; .$$
Show the converse.

(7) Let $f : \mathbb{R} \to \mathbb{R}_+$. Show that if $0 < p < q$, then then the property "$f^q(t)$ is concave on its support" implies "$f^p(t)$ is concave on its support". Passing to the limit $p \to 0$, show that this implies that $\log f(t) : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave (i.e. that f is log-concave).

(8) Show that if A is compact in $\mathbb{R}^n$ then so is its Steiner symmetrization $S_H A$.

(9) Let A be a compact subset of $\mathbb{R}^n$. Show that $S_H A = A$ for all centered hyperplanes H, if and only if A is a centered Euclidean ball.

(10) Let $K \subset \mathbb{R}^n$ be a convex body, and let E be an m-dimensional subspace. Show that the function:
$$E \ni y \mapsto Vol(K \cap (y + E^\perp))^{\frac{1}{n-m}} \; ,$$
is concave on its support.

(11) Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a function so that $f^{1/k}$ is concave on its support, for some integer k, and let E denote an m-dimensional subspace. Denoting:
$$g(y) := \int_{y+E^\perp} f(x)dx \; ,$$
show that the function:
$$E \ni y \mapsto g(y)^{\frac{1}{n+k-m}} \; ,$$
is concave on its support (Hint: construct an appropriate convex set in $\mathbb{R}^{n+k}$). Taking $k \to \infty$, deduce (what we have already seen in class) that when f is log-concave, then so is its marginal g.

(12) Show that the isoperimetric inequality on $\mathbb{R}^n$:
$$|\partial A| \geq n \, |B_2^n|^{1/n} \, |A|^{(n-1)/n} \; ,$$
implies the Brunn-Minkowski inequality when one of the sets is a Euclidean ball $D = rB_2^n$:
$$\mathrm{Vol}(A + D)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(D)^{1/n} \; ,$$
(say for sets A with smooth boundary).

(13) Let $K = -K$ be an origin-symmetric convex body in $\mathbb{R}^n$ and E a linear subspace of dimension k. Using exercise 10, prove the Rogers–Shephard inequality:

$$|K| \leq \left|K \cap E^{\perp}\right| |Proj_E K| \leq \binom{n}{k} |K| \ .$$

Here $|\cdot|$ denotes the volume in the corresponding dimension (n, $n-k$, k and n, respectively).

(14) Prove that $\mathbb{R}_+ \ni r \mapsto Vol(K \cap rD_n)^{1/n}$ is a concave function, for any convex K in $\mathbb{R}^n$.

(15) Recall that Steiner's formula states that for all $t \geq 0$:

$$Vol(K + tB_2^n) = \sum_{i=0}^{n} \binom{n}{i} W_{n-i}(K) t^i \ .$$

Let K be a smooth convex body. Prove using Steiner's formula and integration in polar coordinates that $W_1(K) = Vol(B_2^n) W(K)$, where recall:

$$W(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) \ .$$

Here σ is the uniform probability measure on S^{n-1} and $h_K(\theta) = \sup\{\langle x, \theta\rangle \ ; \ x \in K\}$ is the support function of K.

(16) Calculate the volume of B_p^n, the unit-ball of ℓ_p^n, by integrating the measure $\exp(-\|x\|_{\ell_p^n}^p)dx$ on $\mathbb{R}^n$.

(17) Consider the metric-measure space (S^n, d, μ_{S^n}), where $\mu = \mu_{S^n}$ it the corresponding Haar probability measure, and d is the geodesic distance on S^n. Let $f : S^n \to \mathbb{R}$ be a 1-Lipschitz function, let $m(f)$ denote its median, and define $E(f) = \int f d\mu$. Recall that we've shown that:

$$\mu(x \in S^n \ ; \ f(x) - m(f) \geq r) \leq \sqrt{\frac{\pi}{8}} \exp\left(-\frac{n-1}{2} r^2\right)$$

(a) Show that $|E(f) - m(f)| \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
(b) Show that $0 \leq \sqrt{E(f^2)} - E(|f|) \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
(c) Deduce concentration around $E(f)$, and if $f \geq 0$, also around $\sqrt{E(f^2)}$. In other words, show that:

$$\mu(x \in S^n \ ; \ |f(x) - A_f| \geq r) \leq C \exp\left(-\frac{n-1}{2} r^2\right) \ ,$$

where A_f is either $E(f)$, and when $f \geq 0$, also $\sqrt{E(f^2)}$, for some constant $C > 0$.

(d) Show that for a general function $f : S^n \to \mathbb{R}$:

$$\mu(x \in S^n \; ; \; f(x) - m(f) \geq \omega_f(r)) \leq \sqrt{\frac{\pi}{8}} \exp\left(-\frac{n-1}{2} r^2\right) \; ,$$

where $\omega_f(r) = \sup \{|f(x) - f(y)| \; ; \; d(x,y) \leq r\}$ denotes the modulus of continuity of f.

(18) Show that the Gaussian isoperimetric inequality on $(\mathbb{R}^n, |\cdot|, \gamma_n)$, where γ_n denotes the standard Gaussian measure on $\mathbb{R}^n$, namely:

$$\gamma_n(A) = \gamma_n(H) \quad \Rightarrow \quad \gamma_n^+(A) \geq \gamma_n^+(H) \; ,$$

implies:

$$\gamma_n(A) = \gamma_n(H) \quad \Rightarrow \quad \gamma_n(A_r) \geq \gamma_n(H_r) \; , \; \forall r > 0 \; .$$

Here H is (any) half-plane, $\gamma_n^+(C)$ denote the Gaussian boundary measure of a Borel set $C \subset \mathbb{R}^n$, and C_r denote the r-extension of C.

(19) Let A denote a $k \times n$ random matrix with i.i.d. standard Gaussian entries. Show that with very high-probability (quantify this!), the map $T : \mathbb{R}^n \to \mathbb{R}^k$ given by $T(x) := \frac{1}{\sqrt{k}} Ax$ is a good "Johnson-Lindenstrauss" map, i.e.:

$$(1 - \varepsilon)|z|_{\mathbb{R}^n} \leq |T(z)|_{\mathbb{R}^k} \leq (1 + \varepsilon)|z|_{\mathbb{R}^n}$$

with very high-probability for a fixed $z \in \mathbb{R}^n$.
Guidance: recall we did the same in class for a random projection onto a k dimensional subspace..

(20) Assume that on a metric-measure (probability) space (Ω, d, μ) we know that:

$$\forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A_r) < K(r) \;\; \forall r > 0 \; .$$

Show that:

$$\forall A \subset \Omega \quad \mu(A) \geq K(\varepsilon_0) \quad \Rightarrow \quad \mu(\Omega \setminus A_{r+\varepsilon_0}) < K(r) \;\; \forall r > 0 \; .$$

Deduce that on S^n, we have the following concentration property:

$$\mu(A) \geq \sqrt{\pi/8} \exp(-\frac{n-1}{2} \varepsilon_0^2) \quad \Rightarrow \quad \mu(S^n \setminus A_{2\varepsilon_0}) \leq \sqrt{\pi/8} \exp(-\frac{n-1}{2} \varepsilon_0^2) \; .$$

(21) Two Point Symmetrization on (S^n, d, Vol): let A be a compact set, let $S_\Phi A$ denote its two-point symmetrization with respect to $\Phi \in S^n$, and let C denote a spherical-cap having the same volume as A. Verify that:
(a) $Vol(S_\Phi(A)) = Vol(A)$
(b) $S_\Phi(A)$ is closed.
(c) $(S_\Phi A)_\varepsilon \subset S_\phi(A_\varepsilon)$.
(d) $S_\Phi(C)$ is another spherical-cap.
(e) The set $T = \{B \subset \mathcal{K}(S^n) \; ; \; Vol(B) = Vol(A) \; , \; Vol(B_\varepsilon) \leq Vol(A_\varepsilon)\}$ is closed in $(\mathcal{K}(S^n), \mathcal{H})$ (see exercise 2 to recall definitions).
(f) The mapping $(\mathcal{K}(S^n), \mathcal{H}) \ni B \mapsto Vol(B \cap C)$ is upper semi-continuous.

(22) Prove Bobkov's 3-point inequality:

$$I\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(\sqrt{I(a)^2+\left(\frac{b-a}{2}\right)^2}+\sqrt{I(b)^2+\left(\frac{b-a}{2}\right)^2}\right) \quad , \ \forall a,b \in [0,1] \ .$$

Here I denotes the Gaussian isoperimetric profile, i.e. $I = \varphi \circ \Phi^{-1}$, where φ denotes the standard one-dimensional Gaussian density, and Φ is its cumulative distribution function.

(23) Formulate and prove the sharp isoperimetric inequality for the space $(\mathbb{R}^k, \|\cdot\|, \gamma_k)$, where γ_k is the k-dimensional standard Gaussian measure and $\|\cdot\|$ is a norm on $\mathbb{R}^k$ having unit ball $K = \{x; \|x\| \leq 1\}$.

(24) Prove that if T is an Ehrhard ℓ-symmetrization, then $T(A)_t \subset T(A_t)$.
Hint: use the Gaussian isoperimetric inequality in $\mathbb{R}^\ell$.

(25) Recall Ehrhard's Inequality (E.I.): for any $A, B \subset \mathbb{R}^k$ and $\lambda \in [0,1]$,

$$\Phi^{-1}(\gamma_k(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma_k(A)) + (1-\lambda)\Phi^{-1}(\gamma_k(B)).$$

(a) Use E.I. in dimension ℓ to show that if A is convex then $T(A)$ remains convex for any Ehrhard ℓ-symmetrization T (we will only need this for $\ell = 1$ below).
(b) Assume we've already proved E.I. for convex sets in dimension 1. Use 1-symmetrizations in $\mathbb{R}^{k+1}$ to prove E.I. for convex sets in $\mathbb{R}^k$ (Hint: recall the proof of Brunn's concavity principle).

(26) This exercise is about the Functional version of Ehrhard's Inequality (F.E.I.): given $f, g, h : \mathbb{R}^k \to [0,1]$ and $\lambda \in [0,1]$, if:

$$\forall x, y \in \mathbb{R}^k \quad \Phi^{-1}(h(\lambda x + (1-\lambda)y)) \geq \lambda\Phi^{-1}(f(x)) + (1-\lambda)\Phi^{-1}(g(y)),$$

then:

$$\Phi^{-1}\left(\int h d\gamma_k\right) \geq \lambda\Phi^{-1}\left(\int f d\gamma_k\right) + (1-\lambda)\Phi^{-1}\left(\int g d\gamma_k\right).$$

(a) Show that F.E.I. in $\mathbb{R}^k$ implies E.I. in $\mathbb{R}^k$.
(b) Show that E.I. in $\mathbb{R}^{k+1}$ implies F.E.I. in $\mathbb{R}^k$.

(27) Given a mm-space (Ω, d, μ), prove a log-Sobolev inequality implies a Spectral-Gap inequality. Specifically, prove that $\lambda_{SG} \geq \rho_{LS}$, by using $f_\varepsilon = 1 + \varepsilon h$ with $\int h d\mu = 0$ in the log-Sobolev inequality, showing that $Ent_\mu(f_\varepsilon^2) = 2\varepsilon^2 \int h^2 d\mu + o(\varepsilon^2)$, and taking the limit as $\varepsilon \to 0$.

(28) Formulate and prove the tensorization property of the Spectral-Gap inequality (hint: recall the tensorization property of Bobkov's inequality seen in class).

(29) Prove the Sobolev-Gagliardo inequalities in Euclidean space $(\mathbb{R}^n, |\cdot|)$ (with non-sharp constant):

$$\||\nabla g|\|_{L^q} \geq \frac{n}{n-1}\frac{c_n}{p}\|g\|_{L^p} \quad \forall \text{ compactly supported Lipschitz } g \ \ ,$$

where $1 \leq q < n$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, and c_n denotes the sharp constant in the Euclidean isoperimetric inequality: $|\partial A| \geq c_n |A|^{\frac{n-1}{n}}$.
Hint: translate the isoperimetric inequality into its functional form $q = 1$ above. Then substitute $g = f^r$, and conclude by applying Hölder's inequality.

(30) Given a metric-measure space (Ω, d, μ), prove that a Gaussian isoperimetric inequality $\mathcal{I} \geq D\mathcal{I}_\gamma$ (here $\mathcal{I}_\gamma$ denotes the Gaussian isoperimetric profile and $D > 0$) implies a log-Sobolev inequality:

$$\frac{D^2}{2}\mathrm{Ent}_\mu(f^2) \leq \int |\nabla f|^2 d\mu \quad \forall \text{ Lipschitz } f \ \ .$$

Hint: assume that f is bounded. Pass to Bobkov's functional version of the Gaussian isoperimetric inequality:

$$\int \sqrt{|\nabla g|^2 + (D\mathcal{I}_\gamma(g))^2} d\mu \geq D\mathcal{I}_\gamma(\int g d\mu) \quad \forall \text{ Lipschitz } g : \Omega \to [0,1] \ \ ,$$

apply to $g = \varepsilon f^2$, take the limit as $\varepsilon \to 0$ and use that $\mathcal{I}_\gamma(v) = \sqrt{2}v\sqrt{\log 1/v}(1+o(v))$ as $v \to 0$.

(31) Recall we saw in class how to use Herbst's method to prove that a log-Sobolev inequality implies sub-Gaussian concentration. Use Herbst's method to show that if the following log-Sobolev inequality holds for some $\rho > 0$:

$$\rho \, Ent_\mu(g) \leq \int \frac{|\nabla g|^2}{g} d\mu,$$

for all nice $g > 0$, and if f is 1-log-Lipschitz: $|\nabla \log f| \leq 1$, then f has comparable $L^p(\mu)$ moments:

$$\|f\|_{L^q(\mu)} \leq \exp(\frac{q-p}{\rho}) \|f\|_{L^p(\mu)} \quad \forall 0 < p < q.$$

(32) Recall that the spherically symmetric decreasing rearrangement of a measurable function $f : \mathbb{R}^n \to \mathbb{R}_+$ is defined as the function $f^* : \mathbb{R}^n \to \mathbb{R}_+$ given by:

$$f^*(x) = \sup\{s \geq 0; Vol\{f \geq s\} \geq Vol(B(0,|x|))\},$$

where $B(0,a)$ denotes the ball centered at the origin of radius a. Show that:

$$Vol\{f^* \geq t\} = Vol\{f \geq t\} \quad \forall t \geq 0.$$

(33) Prove the spherical Faber-Krahn inequality: for any $\Omega \subset S^n$, one has the following inequality for the first eigenvalue of the Laplacian with zero Dirichlet boundary conditions on the corresponding domains:

$$\lambda_1^D(\Omega) \geq \lambda_1^D(\Omega^*),$$

where Ω^* denotes the geodesic ball having the same volume as Ω.

(34) Given a reversible semi-group P_t with respect to an invariant measure μ on $\mathbb{R}^n$, the heat-kernel $k_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is defined to be the function $k_t(x, y)$ (if it exists) so that $P_t f(x) = \int f(y) k_t(x, y) d\mu(y)$ for all f and $x \in \mathbb{R}^n$.

(a) Let γ denote the standard Gaussian probability measure on $\mathbb{R}^n$. Verify that the Mehler formula for the Ornstein-Uhlenbeck (O-U) semi-group:

$$P_t f(x) = \int f(\exp(-t)x + \sqrt{1 - \exp(-2t)}y) d\gamma(y),$$

indeed solves the diffusion equation:

$$\frac{d}{dt} P_t f = L P_t f,$$

for the O-U generator $Lf = \Delta f(x) - \langle x, \nabla f(x) \rangle$.

(b) Derive a formula for the O-U heat-kernel $k_t(x, y)$.

(c) For the standard heat equation on $\mathbb{R}^n$, $\frac{d}{dt} P_t f = \Delta P_t f$ with respect to the invariant Lebesgue measure dx, verify that the heat-kernel is given by:

$$k_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{(x - y)^2}{4t}).$$

(35) Recall the Bochner formula:

$$\frac{1}{2} \Delta_g |\nabla f|^2 = \langle \nabla f, \nabla \Delta_g f \rangle + \left\langle \nabla_g^2 f, \nabla_g^2 f \right\rangle + Ric_g(\nabla f, \nabla f),$$

where $\nabla_g^2 f$ denotes the Hessian of f, $\Delta_g f$ denotes its Laplacian (the trace of $\nabla_g^2 f$), and for two matrices A, B, $\langle A, B \rangle = tr(A^T B)$. Here Ric_g denotes the Ricci curvature – the only thing you need to know is that the condition $Ric_g \geq \lambda g$ means $Ric_g(\nabla f, \nabla f) \geq \lambda |\nabla f|^2$.

Use it to show the Lichnerowicz estimate: On a complete connected n-dimensional compact manifold (M^n, g, vol_g), if $Ric_g \geq \lambda g > 0$, then $\lambda_{SG}(M, g) \geq \frac{n}{n-1} \lambda$, i.e.

$$\int_M f d\mathrm{vol}_g = 0 \;\Rightarrow\; \frac{n}{n-1} \lambda \int f^2 d\mathrm{vol}_g \leq \int |\nabla f|^2 d\mathrm{vol}_g.$$

Guidance: you may use that the Laplacian $-\Delta_g$ has discrete spectrum and in particular an eigenfunction corresponding to the first non-trivial eigenvalue λ_{SG}. All integration by parts work exactly as in $\mathbb{R}^n$ (and the manifold has no boundary, so there is no boundary term). Repeat the proof we saw in class of the generalized Lichnerowicz estimate:

(1) $$Ric_{g,\mu} \geq \lambda g > 0 \Rightarrow \lambda_{SG}(M, g, \mu) \geq \lambda,$$

and note that a more delicate analysis gains a factor of $\frac{n}{n-1}$.

(36) Recall that on a weighted manifold (M, g, μ), the Γ-operators are constructed as follows:

$$\Gamma_{i+1}(f, h) = \frac{1}{2}\left(L\Gamma_i(f, h) - \Gamma_i(Lf, h) - \Gamma_i(f, Lh)\right)$$

with $\Gamma_0(f, h) = f \cdot h$. In particular, $\Gamma_1(f, h) = g(\nabla f, \nabla h)$ and $\Gamma_2(f, h) = tr_g(Hess_g f \cdot Hess_g h) + Ric_{g,\mu}(\nabla f, \nabla h)$. Recall that we set $\Gamma_i(f) := \Gamma_i(f, f)$, and write that $\Gamma_2 \geq \lambda\Gamma_1$ if this holds whenever evaluating any function f. Recall that L is the generalized Laplacian, and that the heat semi-group P_t solves $\frac{d}{dt}P_t f = LP_t f$.

(a) Show that the assumption $\Gamma_2 \geq \lambda\Gamma_1$ for $\lambda \in \mathbb{R}$ implies the following two pointwise inequalities:

(2) $$\frac{\exp(2\lambda t) - 1}{\lambda}|\nabla P_t f|^2 \leq P_t(f^2) - P_t(f)^2 \leq \frac{1 - \exp(-2\lambda t)}{\lambda}P_t(|\nabla f|^2).$$

(with the interpretation when $\lambda = 0$ in the limiting sense, e.g. $\frac{\exp(2\lambda t)-1}{\lambda}|_{\lambda=0} = 2t$). Guidance: consider the function $P_s((P_{t-s}f)^2)$, and repeat the proof we saw in class for showing the inequality between left and right expressions:

(3) $$|\nabla P_t f|^2 \leq \exp(-2\lambda t)P_t(|\nabla f|^2).$$

(b) Deduce from part (a) an alternative proof of the generalized Lichnerowicz estimate (1) which we saw in class .

(c) Show that any of the three inequalities appearing in (2) and (3) above are in fact equivalent to the property that $\Gamma_2 \geq \lambda\Gamma_1$.