An Intuitive Introduction to Ricci Curvature

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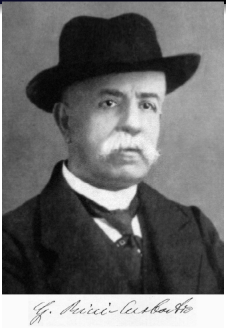
Ricci Curvature and its Applications



Ricci Curvature
(after Gregorio Ricci-Curbastro 1853–1925):

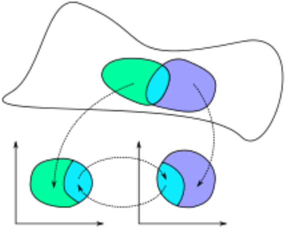
- General Relativity (Einstein, 1915):

$$\mathrm{Ric}_{\alpha,\beta} + (\Lambda - \frac{1}{2} S) g_{\alpha,\beta} = \frac{8\pi G}{c^4} T_{\alpha,\beta}.$$

- Comparison Theorems in Riemannian Geometry.
- Poincaré & Geometrization conjectures via Ricci flow (Hamilton '81, Perelman '02).
- Notion extended to more general settings (Bakry–Émery '85, Lott–Sturm–Villani '04).

Differentiable Manifold M

Abstract definition (charts, transition maps, Hausdorff, σ-compact).



Whitney (1936): M^n can always be embedded as smooth n-dimensional submanifold of $\mathbb{R}^N$ ($N = 2n$).

$\forall p \in M$, $\exists M_p \subset M$, $\exists U_p \subset \mathbb{R}^n$, $\exists$ diffeo $F_p : U_p \to M_p$ s.t. $F_p(0) = p$.

Local coordinates:

$$U_p \ni \vec{x} = (x^1, \ldots, x^n) \mapsto (F_p^1, \ldots, F_p^N) = \vec{F}_p \in M_p.$$

Identify: $U_p \leftrightarrow M_p$ via F_p,
e.g. x^1 on U_p identified with $x^1 \circ F_p^{-1}$ on M_p.

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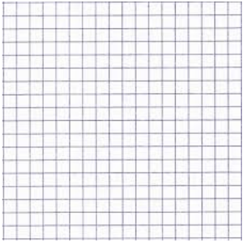
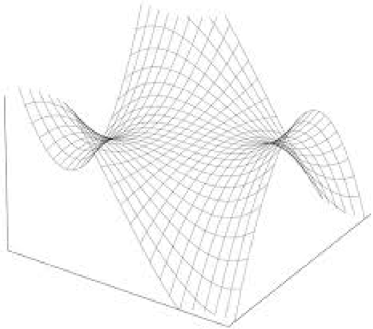


$\longrightarrow$



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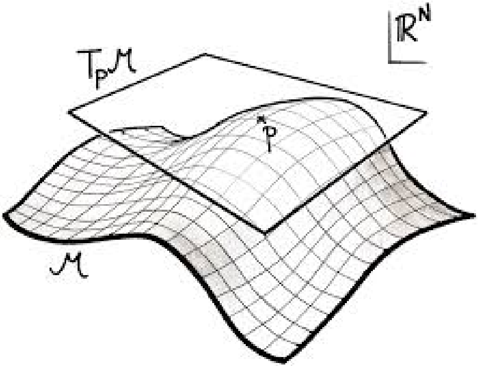
Tangent Space T_pM

T_pM = Tangent space to M at p (abstract def or $p \in M \subset \mathbb{R}^N$).

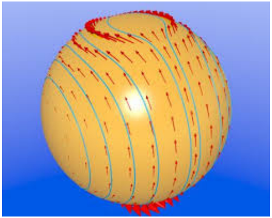


$T_pM \hookrightarrow \mathbb{R}^N$ n-dimensional linear subspace spanned by:

$$\partial_i = \frac{\partial}{\partial x^i} := \frac{\partial \vec{F}_p}{\partial x^i} \ , \ i = 1, \ldots, n.$$

Vector Field

Vector field $V \in C^\infty(TM)$ = smooth selection $M \ni p \mapsto V(p) \in T_pM$.



Smooth = do it locally in $U_p \subset \mathbb{R}^n$, identify via $d_xF_p : T_xU_p \to T_{F_p(x)}M$.

Local coordinates: $\frac{\partial}{\partial x^i}$ are linearly independent in M_p (as F_p diffeo.):

$$V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i} = V^i \frac{\partial}{\partial x^i} \;;\; \text{Identify } V \leftrightarrow V^\alpha .$$

Remark: any $V \neq 0$ is locally $\frac{\partial}{\partial x^1}$, but not any (linearly independent) V_1, V_2 are locally $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$ (iff $[V_1, V_2] = 0$ since $\frac{\partial^2 F_p}{\partial x^1 \partial x^2} = \frac{\partial^2 F_p}{\partial x^2 \partial x^1}$).
Hence, we will only consider tuples of coordinate vector fields.

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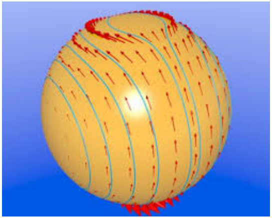

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Cotangent Space T_p^*M

$T_p^*M =$ Cotangent space to M at p = linear dual to T_pM.
Elements called co(tangent)-vectors.

Co-vector field (a.k.a. differential 1-form) $w \in C^\infty(T^*M)$:

$$\forall V \in C^\infty(TM) \quad w(V) \in C^\infty(M).$$

Local coordinates: $\frac{\partial}{\partial x^i}$ basis to TM_p, dx^i dual basis to T^*M_p:

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j \; .$$

$$w = w_i dx^i \; , \; w_i := w\left(\frac{\partial}{\partial x^i}\right) \; ; \text{ Identify } w \leftrightarrow w_\beta .$$

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Tensors

(κ,ℓ)-tensor = multilinear map $Q : (T_p^* M)^{\otimes \kappa} \otimes (T_p M)^{\otimes \ell} \to \mathcal{F}$.
Examples: vectors are $(1,0)$ tensors, co-vectors are $(0,1)$ tensors.

Tensor field = smoothly varying selection

$$M \ni p \mapsto Q(p) \in \mathrm{Lin}((T_p^* M)^{\otimes \kappa} \otimes (T_p M)^{\otimes \ell} \to \mathcal{F}).$$

Local coordinates:

$$Q = Q^{i_1,\ldots,i_\kappa}_{j_1,\ldots,j_\ell} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_\kappa}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_\ell}$$

$$Q^{i_1,\ldots,i_\kappa}_{j_1,\ldots,j_\ell} := Q(dx^{i_1},\ldots,dx^{i_\kappa},\frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_\ell}}) \text{ , identify } Q \leftrightarrow Q^{\alpha_1,\ldots,\alpha_\kappa}_{\beta_1,\ldots,\beta_\ell}.$$

Remark: κ-contravariant / ℓ-covariant under change of coordinates:

$$(Q_y)^b_a = (Q_x)^j_i \frac{dx^i}{dy^a}\frac{dy^b}{dx^j}.$$

Riemannian Manifold (M^n, g)

(M^n, g) = Differentiable manifold M^n + Riemannian metric g.

$g_{\alpha,\beta}$ = symmetric positive-def $(0,2)$-tensor field.
Locally: $g = g_{i,j} dx^i \otimes dx^j$ and $g(u,v) = g_{i,j} u^i v^j \quad \forall u, v \in T_p M$.

Meaning: smoothly varying scalar product on $T_p M$:

$$|v| = \sqrt{g(v,v)} \text{ length }, \ \langle u, v \rangle = g(u,v) \text{ angle.}$$

Nash ('54-'56): Abstract (M^n, g) can always be (globally) isometrically embedded in $(\mathbb{R}^N, \langle \cdot, \cdot \rangle)$, $N \gg n$:

$$g_{i,j} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\langle \frac{\partial F_p}{\partial x^i}, \frac{\partial F_p}{\partial x^j} \right\rangle_{\mathbb{R}^N} \text{ (1st-fundamental form).}$$

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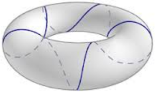
1. Geodesic distance d on M

Geodesic distance (metric):
$d(x,y) := \inf \left\{ \int_0^1 |\gamma'(t)| \, dt \; ; \gamma : [0,1] \to M, \gamma(0) = x, \gamma(1) = y \right\}.$
Minimizers of distance always exist locally.
Geodesics = paths which locally minimize distance.



Equivalently $d(x,y)^2 = \inf \left\{ \int_0^1 |\gamma'(t)|^2 dt \right\}$
(uniqueness under reparametrization $t = t(s)$).

Euler-Lagrange eqns yield 2nd order system
of ODEs. Given initial $\gamma(0) = p, \gamma'(0) = v \in T_pM$
$\Rightarrow \; \exists$ local solution $\gamma : [0, t_0) \to M$;
if $\exists \gamma(1)$, we obtain the exponential map:

$T_pM \supset Dom(exp_p) \ni v \mapsto \exp_p(v) := \gamma(1) \in M.$

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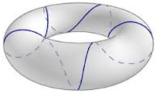

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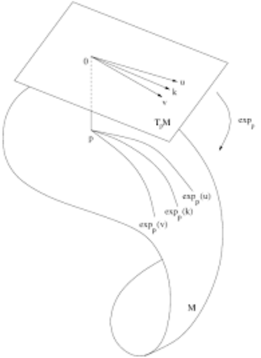

Radius of Injectivity

exponential map: $T_pM \supset Dom(exp_p) \ni v \mapsto \exp_p(v) \in M$.

Injectivity radius at $p \in M$:

$$inj_p := \sup \left\{ r > 0; \exp_p \text{ is injective on } B_{T_pM}(0,r) \subset Dom(\exp_p) \right\}.$$

Facts: $inj_p > 0$ and $\exp_p : B_{T_pM}(0, inj_p) \to B_M(p, inj_p)$ is 1-1 diffeo.

Injectivity radius of M:

$$inj(M) := \inf_{p \in M} inj_p.$$

Relevance: if $inj(M_t) \to 0$ then possibly singularities being formed.

Identify $T_pM \leftrightarrow T_p^*M$ by $u^\flat(v) = g(u,v) \;\; \forall u, v \in T_pM$.
Locally: if $u^\flat = u_i dx^i$, $u_i v^i = g_{i,j} u^j v^i$, or in other words, $u_i = g_{i,j} u^j$.

Similarly, $w(v) = g(w^\sharp, v) \;\; \forall w \in T_p^*M, v \in T_pM$.
Locally: if $w^\sharp = w^i \frac{\partial}{\partial x^i}$, $w_i v^i = g_{i,j} w^j v^i \Rightarrow w_i = g_{i,j} w^j \Rightarrow w^k = g^{k,i} w_i$,
where $g^{k,i}$ denotes g^{-1} in local coordinates, i.e. $g^{k,i} g_{i,j} = \delta_j^k$.

Raising and lower indices by contracting with g allows changing tensor type $(\kappa, \ell) \leftrightarrow (\mu, \nu)$ if $\kappa + \ell = \mu + \nu$. For example:

$$Q_a{}^{b,c,d} = g^{b,i} Q_{a,i}{}^{c,d}.$$

3. Riemannian Volume Measure Vol_M

Riemannian volume measure is $\mathrm{Vol}_M = \sqrt{det(g_{i,j})} dx^1 \ldots dx^n$ (replace $dx^1 \ldots dx^n$ with $dx^1 \wedge \ldots \wedge dx^n$ to get volume form).

Interpretation: Vol_M is the induced Lebesgue measure on $M \hookrightarrow \mathbb{R}^N$.
Since $F_p : \mathbb{R}^n \supset U_p \to M_p \subset \mathbb{R}^N$, we have:

$$\mathrm{Jac}(F_p) = \sqrt{\det(dF_p^t dF_p)} \; , \; (dF_p^t dF_p)_{i,j} = \left\langle \frac{\partial F_p}{\partial x^i}, \frac{\partial F_p}{\partial x^j} \right\rangle_{\mathbb{R}^N} = g_{i,j}.$$

Integration: integrate $f : M \to \mathbb{R}$ in local coordinates in M_α, and then declare $\int_M f \, d\mathrm{Vol}_M = \sum_\alpha \int_{M_\alpha} f p_\alpha d\mathrm{Vol}_M$, where p_α = partition of unity.

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Affine Connection, Covariant Derivative

Derivative of function in direction of $v \in T_pM$:

$$\gamma(t) := \exp_p(tv)\ ,\ \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} = df(\gamma'(0)) = \nabla_v f.$$

What about derivative of vector field X in direction $v \in T_pM$?
Need to connect between tangent spaces along a path $\gamma : [0,1] \to M$.
“Connection". Transporting vector along γ called “Parallel Transport".

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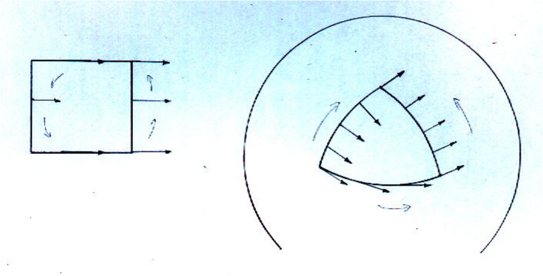

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$$0 = \text{“}\frac{d}{dt}\text{”} X(\gamma(t)) = \nabla_{\gamma'(t)} X \;,\; X(\gamma(0)) = X_0.$$

So need to specify $\nabla_v X, \forall v \in T_pM, X \in TM$. By linearity in v and additivity + Leibnitz rule in X, enough to specify $\nabla_{\partial_i} \partial_j = \Gamma^k_{i,j} \partial_k$.
Yields system of 1st order linear ODEs $\Rightarrow$ $\exists$ solution (on entire γ).

∇ called “Covariant Derivative": X is $(1,0)$ tensor, and ∇X is $(1,1)$.

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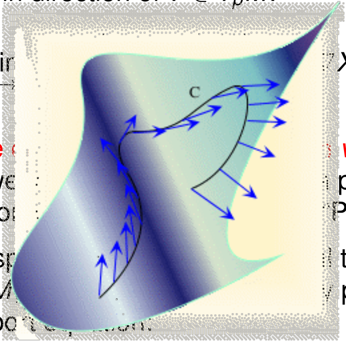

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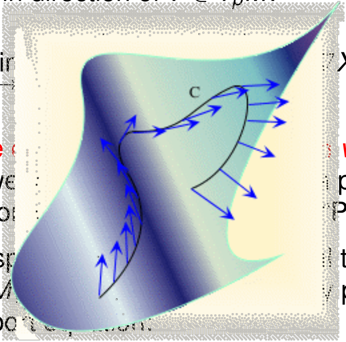

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4. Levi-Civita Connection

Choice of affine connection ($\Gamma^k_{i,j}$) may be arbitrary.

We would like a connection which is:

- Compatible with metric g = parallel transport is local isometry (preserves lengths and angles):

$$\nabla_v g(X,Y) = g(\nabla_v X, Y) + g(X, \nabla_v Y) \quad \text{(i.e. } \nabla g = 0\text{)}.$$

- Torsion free (= doesn't spin vectors unnecessarily):

$$\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i \quad \forall i,j.$$

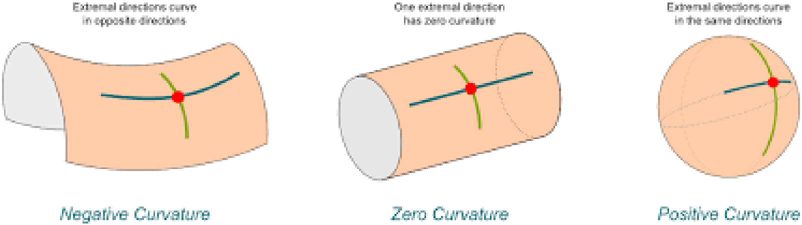
Levi-Civita ('29): $\exists$ unique connection / covariant derivative as above.

$$\left[\Gamma^k_{i,j} = \frac{1}{2} g^{k,a} \left(\frac{\partial}{\partial x^i} g_{a,j} + \frac{\partial}{\partial x^j} g_{i,a} - \frac{\partial}{\partial x^k} g_{i,j} \right) \right].$$

5. Sectional Curvature - Definition

Assume (M, g) is 2-D surface, locally isometrically embedded in $\mathbb{R}^3$.

Let k_1, k_2 be the principle curvatures at $p \in M$ = eigenvalues of Hess f where $f : T_pM \to \mathbb{R}$ s.t. $M = \text{graph}(f)$ (locally near p).



Gauss Theorema Egregium (1827): Gauss Curvature $K = k_1 \cdot k_2$ is intrinsic, does not depend on isometric embedding in $\mathbb{R}^N$, only on g.

Examples: $K(\mathbb{S}^2(r)) \equiv 1/r^2$, $K(\text{Cylinder}/\mathbb{R}^2) \equiv 0$, $K(\mathbb{H}^2) = -1$.

Def (Riemann, 1854): Let (M^n, g), $E \hookrightarrow T_pM$ a 2-D plane.

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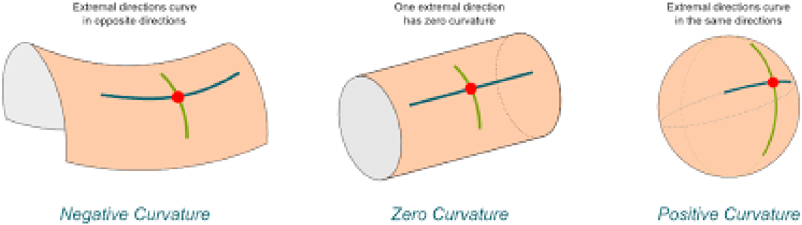

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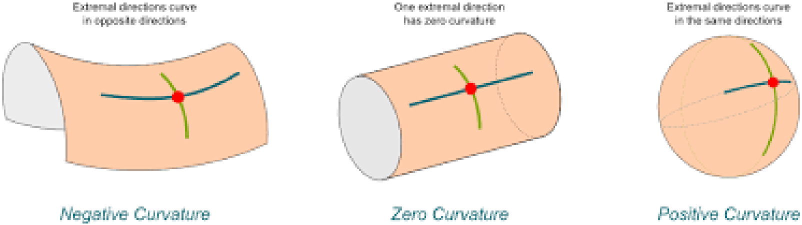

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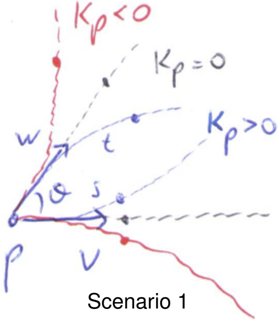
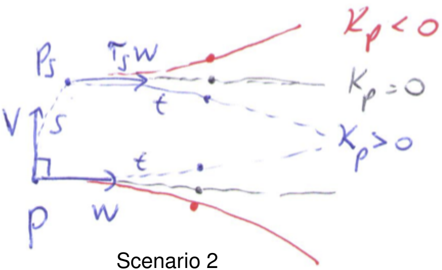
5. Sectional Curvature - Geometric Meaning

Let $v, w \in T_pM$, $|v| = |w| = 1$.



Scenario 1



Scenario 2

1. $d^2(\exp_p(sv), \exp_p(tw)) = s^2+t^2-2st\cos(\theta)-\frac{K_p(v\wedge w)}{3}s^2t^2\sin^2(\theta)+O((s^2+t^2)^{5/2})$.

2. Let $p_s = \exp_p(sv)$, let $\tau_s w =$ parallel transport of w along geodesic $p \to p_s$. Then:

$$d(\exp_p(tw), \exp_{p_s}(t\,\tau_s w)) = s\left(1-\frac{K_p(v\wedge w)}{2}t^2+O(t^3+t^2 s)\right).$$

$\Rightarrow$ Sectional curvature controls distance distortion of geodesics.

Riemann Curvature Tensor = $(0,4)$ tensor R constructed from $\{K_p(E)\,;\,$2-D $E \hookrightarrow T_pM\}$ by polarization such that:

$\forall$ orthonormal $u, v \in T_pM \quad R_p(u,v,u,v) = K_p(u \wedge v) \; + \;$ symmetries.

e.g. $R_p(u,v,u,v) = K_p(u \wedge v)|u \wedge v|^2$, $|w_1 \wedge w_2|^2 = \det(\langle w_i, w_j \rangle)$.
$R_p(u,v,w,z) = \ldots$ (18 terms) $\ldots$

$$R = R_{i,j,k,\ell}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell \quad , \quad R_{i,j,k,\ell} = R(\partial_i, \partial_j, \partial_k, \partial_\ell).$$

Differential Definition of (1,3) version (Gauss Theorema Egregium):

$$[\nabla_{\partial_j}, \nabla_{\partial_i}]\partial_k = \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{\partial_i}\nabla_{\partial_j}\partial_k = R_{i,j,k}{}^{\ell}\partial_\ell.$$

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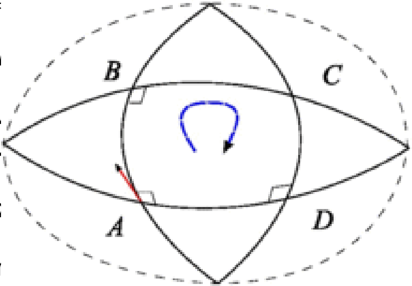

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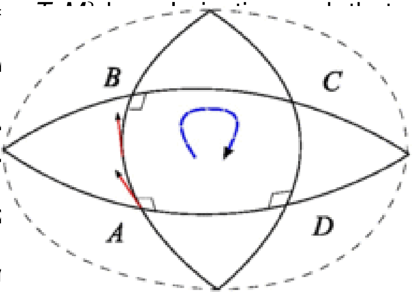

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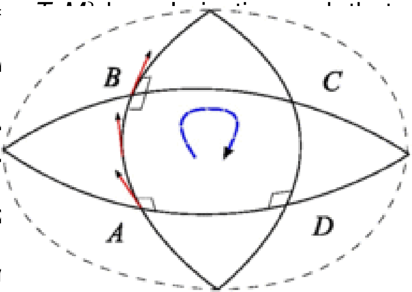

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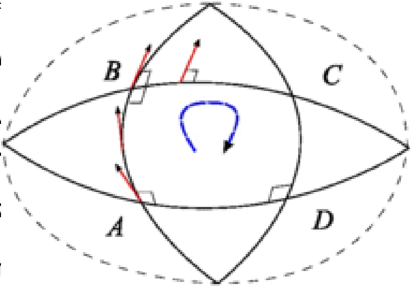

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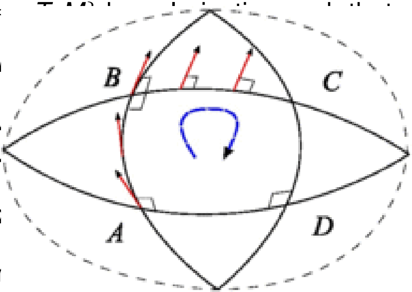

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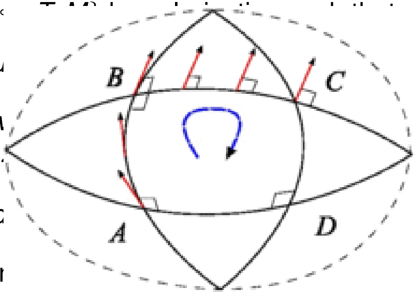

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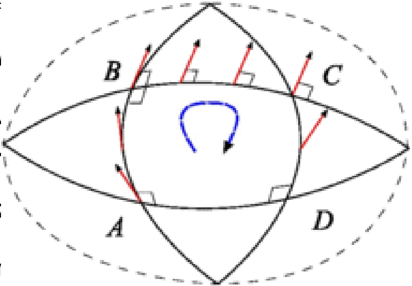

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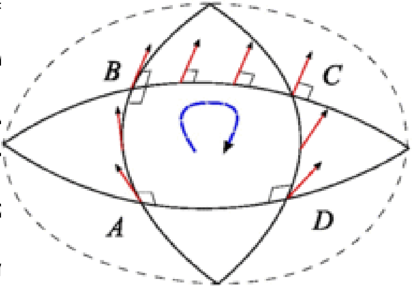

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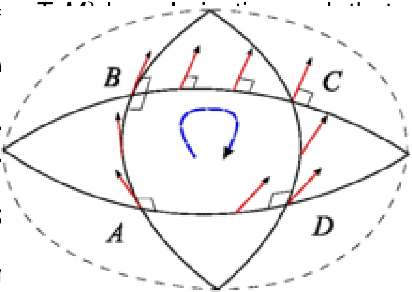

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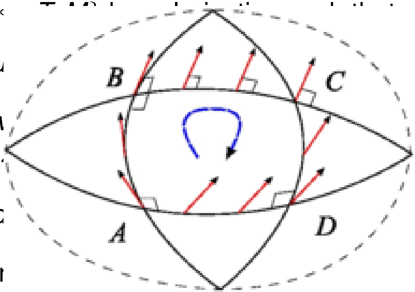

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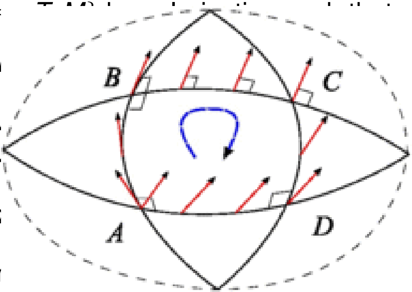

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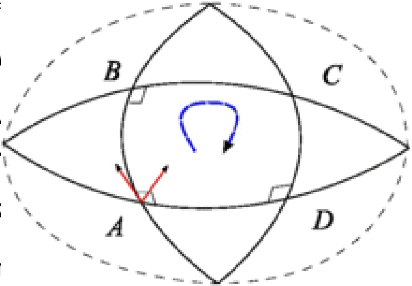
$$\nabla_{\partial_j}\nabla_{\partial_i} Z - \nabla_{\partial_i}\nabla_{\partial_j} Z = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\Big|_{s=t=0} \tau_{s\partial_j}^{-1}\tau_{t\partial_i}^{-1}\tau_{s\partial_j}\tau_{t\partial_i} Z .$$

Riemann Curvature Tensor = $(0,4)$ tensor R constructed from $\{K_p(E)\ ;\ \text{2-D } E$



$\forall$ orthonormal u … $)\ +$ symmetries.

e.g. $R_p(u,v,u,v$ … $\det(\langle w_i, w_j\rangle)$.

$R_p(u,v,w,$

$R = R_{i,j,k,\ell}\, d$ … $R(\partial_i,\partial_j,\partial_k,\partial_\ell)$.

Differential Defi … ema Egregium):

$$[\nabla_{\partial_j},\nabla_{\partial_i}]\partial_k = \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{\partial_i}\nabla_{\partial_j}\partial_k = R_{i,j,k}{}^{\ell}\,\partial_\ell .$$

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6. Ricci Curvature - Definition

Geometrically: given $v \in T_pM$, $|v| = 1$, complete to orthonormal basis $v, w_1, \ldots, w_{n-1}$, and define:

$$\textit{Ricci curvature:} \quad \mathrm{Ric}_p(v) := \sum_{k=1}^{n-1} \mathrm{Sect}_p(v \wedge w_k).$$

$\mathrm{Ric}(u, v)$ = symmetric $(0, 2)$ Ricci curvature tensor obtained by polarization s.t. $\mathrm{Ric}(v, v) = \mathrm{Ric}(v)$ for all $|v| = 1$.

Algebraically: $\mathrm{Ric}_{i,j} = \mathrm{tr}_g R_{i,*,j,*} = g^{k,\ell} R_{i,k,j,\ell} = R_{i,k,j}{}^{k}$ (non-trivial tr):

$$\mathrm{Ric}(u, v) = \sum_{k=1}^{n} R(u, e_k, v, e_k) \;\; \forall \text{ orthonormal basis } e_k \;\; (R(v, v, *, *) = 0).$$

(symmetries of R yield equivalence + symmetry of Ric).

Example: $\mathrm{Ric}_{\mathbb{S}^n}(v, v) = (n-1)\,|v|^2$ i.e. $\mathrm{Ric}_{\mathbb{S}^n} = (n-1)g$.

Scalar Curvature: $S = \mathrm{tr}_g \mathrm{Ric} = g^{i,j}\mathrm{Ric}_{i,j} = \mathrm{Ric}_i{}^{i}$ (scalar function).

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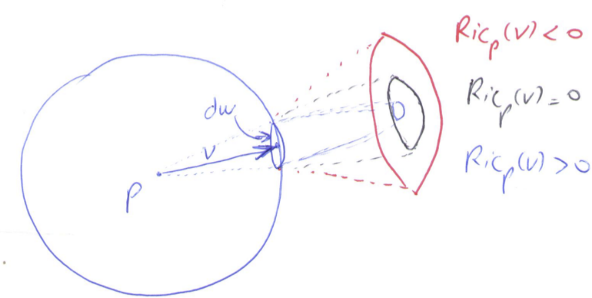
6. Ricci Curvature - Geometric Meaning



$$v \in d\omega \subset S^{n-1}(T_pM) \;\Rightarrow\; \mathrm{vol}_{n-1}(\exp_p(\varepsilon d\omega)) = \mathrm{vol}_{n-1}(d\omega)\varepsilon^{n-1}\left(1 - \frac{\mathrm{Ric}_p(v)}{6}\varepsilon^2 + O(\varepsilon^3)\right).$$

$\Rightarrow$ $\mathrm{Ric}_p(v)$ controls distortion of vol_{n-1} in direction $v \in T_pM$. Intuition:

- $det(I + \varepsilon A) = 1 + \varepsilon tr(A) + o(\varepsilon) =$ averaging $n-1$ distance distortions.
- In normal (geodesic) coordinates (i.e. $x = x^i e_i \in T_pM$ and $F_p(x) = \exp_p(x)$):
  $g_{i,j} = \delta_{i,j} - \frac{1}{3}R_{p,i,q,j}x^p x^q + O(|x|^3) \;\Rightarrow\; det(g_{i,j}) = 1 - \frac{1}{3}\mathrm{Ric}_{p,q}x^p x^q + O(|x|^3)$.

Applications of Ricci lower bounds

Given complete connected (M^n, g) with $\mathrm{Ric} \geq K(n-1)g$, many comparison thms to model space (M^n_K, g_K) with $\mathrm{Sect} \equiv K$:

- Surface Area vs. Volume of geodesic balls (Bishop–Gromov).
- Isoperimetric inequalities (Lévy–Gromov).
- First eigenvalue of Laplacian (Lichnerowicz / Cheng).
- Diameter $\leq \pi/\sqrt{K}$ if $K > 0$ (Bonnet–Meyers).
- Parabolic estimates comparison (Li–Yau, Bakry–Émery).
- Heat-kernel comparison (Cheeger–Yau).
- many, many more ...

Typically: comparison inqs reversed if (M^n, g) satisfies $\mathrm{Sect} \leq K$!

Example: Bishop–Gromov Theorem

Comparison between complete (M^n, g) and (M^n_K, g_K), $K \in \mathbb{R}$.

Set $V_*(R) = Vol(B(p,R))$ and $S_*(R) = Vol_{n-1}(\partial B(p,R))$ in (M^n_*, g_*).

Gromov ('80):

$$\text{Ric} \geq K(n-1)g \quad \Rightarrow \quad \frac{S(R)}{V(R)} \leq \frac{S_K(R)}{V_K(R)} \Leftrightarrow \frac{V(R)}{V(r)} \leq \frac{V_K(R)}{V_K(r)} \quad \forall 0 < r < R.$$

Letting $r \to 0$, obtain Bishop's Thm ('63):

$$\text{Ric} \geq K(n-1)g \quad \Rightarrow \quad V(R) \leq V_K(R) \quad \forall R > 0.$$

However, to obtain reverse comparison (Günther '60), require:

$$\text{Sect} \leq K \quad \Rightarrow \quad \frac{V(R)}{V(r)} \geq \frac{V_K(R)}{V_K(r)} \quad \forall 0 < r < R \leq \text{inj}_p.$$

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Images courtesy of:

- www.wikipedia.org / www.wikimedia.org.
- ww.science4all.org.
- www.zweigmedia.com.
- JefferyWinkler.com.
- Blagoje Oblak, arXiv:1508.00920.
- Paulo H.F Reimberg et al., arXiv:1211.5114.
- Paul M. Alsing et al., arXiv:1308.4148.

If $\{t \mapsto \gamma_s(t)\}_s$ is a family of geodesics, then its variation $\xi(t) := \frac{\partial \gamma}{\partial s}|_{s=0}$ is called a Jacobi vector field along γ_0.

It satisfies the Jacobi Equation:

$$(\frac{\nabla}{dt})^2 \xi + R(\gamma'(t), \xi)\gamma'(t) = 0.$$

Proof.

Commuting covariant derivatives twice, we obtain by definition of R:

$$\frac{\nabla}{dt}\frac{\nabla}{dt}\xi = \frac{\nabla}{dt}\frac{\nabla}{dt}\frac{\nabla}{ds}\gamma =_{(*)} \frac{\nabla}{dt}\frac{\nabla}{ds}\frac{\nabla}{dt}\gamma = \frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\nabla}{dt}\gamma + R(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t})\frac{\partial \gamma}{\partial t},$$

where (*) follows since ∇ is torsion-free:

$$\nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} - \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} = [\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial s}] = 0.$$

Conclude using $(\frac{\nabla}{dt})^2 \gamma = \nabla_{\gamma'(t)}\gamma'(t) = 0$ since γ is geodesic.

Bonus Material - 2. Initial Conditions

Given family of $n-1$ Jacobi fields along unit-speed geodesic $t \mapsto \gamma(t)$, arrange them as columns in $(n-1) \times (n-1)$ matrix $A(t)$. Then column-wise:

$$(\frac{\nabla}{dt})^2 A(t) + R(t)A(t) = 0 \; , \; R(t) := R(\gamma'(t), \cdot, \gamma'(t), \cdot).$$

Typically $A(t) = d_x F(x,t)$, $S \times \mathbb{R} \ni (x,t) \mapsto F(x,t) := \exp_{p(x)}(t\nu(x))$.
Examples:

1. Exponential map from point p.
   $S = S^{n-1}(T_p M)$, $p(x) \equiv p$, $\nu(x) = x$, and:

   $$A(0) = 0 \; , \; \frac{\nabla}{dt} A(0) = \mathrm{II}_{S^{n-1}}(x) = Id.$$

2. Normal map from oriented hypersurface $S \subset M$.
   $p(x) = x$, $\nu(x) =$ unit-normal to S at x, and:

   $$A(0) = Id \; , \; \frac{\nabla}{dt} A(0) = \mathrm{II}_S(x) \text{ (second fundamental form).}$$

Bonus Material - 3. Tracing Matrix Riccati Equation

Denoting $U(t) = \frac{\nabla}{dt} A(t) A^{-1}(t)$, easy to check that:

$$(\frac{\nabla}{dt})^2 A(t) + R(t)A(t) = 0 \;\; \Leftrightarrow \;\; \frac{\nabla}{dt} U(t) + U(t)^2 + R(t) = 0,$$

as long as $A(t)$ is invertible (before focal point). Symmetry of $R(t)$ and $U(0)$ ensure symmetry of $U(t)$, and hence $U(t)^2 \geq 0$.
Taking traces and applying Cauchy-Schwarz $\frac{1}{n-1} tr(U)^2 \leq tr(U^2)$:

$$\frac{d}{dt} tr(U(t)) + \frac{tr(U(t))^2}{n-1} + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq 0.$$

But $tr(U) = tr(\frac{\nabla}{dt} A(t) A^{-1}(t)) = \frac{d}{dt} \log det\, A(t) = \frac{d}{dt} \log J(t)$, where J is the Jacobian of $(x, t) \mapsto F(x, t)$. Assuming $\mathrm{Ric} \geq K(n-1)g$, obtain:

$$(\log J)'' + \frac{((\log J)')^2}{n-1} + K(n-1) \leq 0 \;\; \Leftrightarrow \;\; \frac{(J^{\frac{1}{n-1}})''}{J^{\frac{1}{n-1}}} + K \leq 0.$$

By maximum principle, if $J(0) = J_K(0)$, $J'(0) = J_K'(0)$, comparison to model space follows $J \leq J_K$. In fact, obtain $(\log J)' \leq (\log J_K)'$. $\square$