# HIGH DIMENSIONAL CONVEX GEOMETRIC ANALYSIS

(note that the statement of one exercise may be used for the proof of another).

(1) (a) Show that there is a one-to-one correspondence between origin-symmetric convex bodies $K$ in $\mathbb{R}^n$ (i.e. compact convex subsets with non-empty interior), and normed spaces $(\mathbb{R}^n, \|\cdot\|_K)$, given by $\|x\|_K := \inf\{t > 0; x \in tK\}$.
 (b) Show that $K \subset L$ iff $\|x\|_K \geq \|x\|_L$ for all $x \in \mathbb{R}^n$.

(2) Recall that he Banach–Mazur distance between two convex bodies $K, L \subset \mathbb{R}^n$ was defined as:
$$d_{BM}(K,L) := \inf\left\{ab \; ; \; \frac{1}{a}L \subset T(K) \subset bL \; , \; T \in \text{Affine}(\mathbb{R}^n)\right\} \; ,$$
Where $\text{Affine}(\mathbb{R}^n)$ denotes all affine transformations of $\mathbb{R}^n$. Show that when $K = -K$ and $L = -L$, it is enough to just consider linear maps $T \in GL(n)$ above.

(3) Let $\Delta_n$ denote the regular simplex in $\mathbb{R}^n$ with barycenter at the origin. Show that $d_G(\Delta_n, D_n) = n$, where recall, the geometric distance is defined as:
$$d_G(K,L) := \inf\left\{ab \; ; \; \frac{1}{a}L \subset K \subset bL\right\} \; .$$
Hint: use the natural embedding of $\Delta_n \subset \mathbb{R}^{n+1}$ as $\Delta_n := conv(e_1, \ldots, e_{n+1})$.

(4) Recall that $B_\infty^n$ and $B_1^n$ denote the unit-balls of $\ell_\infty^n$ and $\ell_1^n$, respectively.
 (a) Verify that $d_G(B_1^n, D_n) = \sqrt{n}$.
 (b) Verify that $d_G(B_\infty^n, D_n) = \sqrt{n}$.

(5) Let $H$ denote the hyperplane perpendicular to the diagonal in $\mathbb{R}^n$, i.e. $H := (1/\sqrt{n}, \ldots, 1/\sqrt{n})^\perp$. Using the (local) Central-Limit-Theorem, calculate:
$$\lim_{n\to\infty} Vol([-1/2,1/2]^n \cap H) \; .$$
Verify that indeed this value is $\leq \sqrt{2}$. (Here local CLT means that if $\{X_i\}$ are i.i.d. bounded random-variables with $E(X_1) = 0$ and $Var(X_1) = 1$, then the density at $t \in \mathbb{R}$ of $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$ converges to the density of a standard Gaussian random-variable at $t$).

(6) We have proved the Brunn-Minkowski inequality for two $n$-dimensional rectangles with aligned axes. Use this as the basis of an induction, and prove the BM inequality for bodies $K = \cup_{i=1}^N A_i$ and $L = \cup_{j=1}^M B_j$, where $A_i$ and $B_j$ are $n$-dimensional rectangles (with axes aligned with the principle axes of $\mathbb{R}^n$) having mutually disjoint interiors.
Hint: use induction on $N + M$. Use the fact that if $N \geq 2$, one can always find a hyperplane $H$ so that each of the collections $\{A_i \cap H^+\}$ and

$\{A_i \cap H^-\}$ are composed of $\leq N-1$ non-degenerate rectangles. Finally, translate the set $L$ into a favorable position, so that $L \cap H^+$ and $L \cap H^-$ have convenient volumes.

(7) Prove that $det^{1/n}$ is concave on the class of $n \times n$ positive semi-definite matrices:

$$A, B \geq 0 \quad \Rightarrow det(A+B)^{1/n} \geq det(A)^{1/n} + det(B)^{1/n} \ .$$

(8) Recall the *Prékopa–Leindler inequality*: if $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ denote three measurable functions so that for some $\lambda \in (0,1)$:

$$h(\lambda x + (1-\lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda} \quad \forall x, y \in \mathbb{R}^n \ .$$

Then:

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^\lambda \left(\int_{\mathbb{R}^n} g\right)^{1-\lambda} \ .$$

We've seen in class that the 1-D Brunn–Minkowski (BM) inequality implies the 1-D Prékopa–Leindler (PL) inequality. Generalize this implication to arbitrary dimension $n$ using two different methods:

(a) Method 1. First, prove the following geometric-average version of the 1-D PL inequality, by reducing it to the usual (arithmetic-average) 1-D PL inequality: if $a, b, c : \mathbb{R}_+ \to \mathbb{R}_+$ and:

$$c(x^\lambda y^{1-\lambda}) \geq a(x)^\lambda b(y)^{1-\lambda} \quad \forall x, y \in \mathbb{R}_+,$$

Then:

$$\int_{\mathbb{R}_+} c \geq \left(\int_{\mathbb{R}_+} a\right)^\lambda \left(\int_{\mathbb{R}_+} b\right)^{1-\lambda} .$$

Now apply the BM inequality to the level-sets of $h, f, g$, and conclude using the above 1-D geometric-average PL inequality.

(b) Method 2. First prove using the BM inequality that if:

$$h(\lambda x + (1-\lambda)y)^{\frac{1}{k}} \geq \lambda f(x)^{\frac{1}{k}} + (1-\lambda) g(y)^{\frac{1}{k}} \quad \forall x, y \in \mathbb{R}^n$$

for some natural number $k$, then:

$$\left(\int_{\mathbb{R}^n} h\right)^{\frac{1}{n+k}} \geq \lambda \left(\int_{\mathbb{R}^n} f\right)^{\frac{1}{n+k}} + (1-\lambda) \left(\int_{\mathbb{R}^n} g\right)^{\frac{1}{n+k}} \ .$$

(this is a particular case of the Borell / Brascamp-Lieb inequalities). Conclude the PL inequality by taking $k \to \infty$ and using that for $a, b > 0$:

$$\lim_{\varepsilon \to 0} (\lambda a^\varepsilon + (1-\lambda) b^\varepsilon)^{1/\varepsilon} \to a^\lambda b^{1-\lambda} \ .$$

(9) We've seen in class that if $f : \mathbb{R}^n \to \mathbb{R}_+$ is a log-concave function:

$$\forall \lambda \in [0,1] \ \ \forall x, y \in \mathbb{R}^n \quad f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda} \ ,$$

then $\mu = f(x)dx$ is a log-concave measure:

$$\forall A, B \subset \mathbb{R}^n \quad \mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda} \ .$$

Show the converse.

(10) (a) Show that if $f$ is log-concave and integrable, then so are all of its marginals. In other words, show that if $f : \mathbb{R}^{n+m} \to \mathbb{R}_+$ is log-concave and integrable, then so is $h : \mathbb{R}^n \to \mathbb{R}_+$, where $h(x) := \int_{\mathbb{R}^m} f(x,y)dy$.

(b) Deduce from part (a) that if $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are log-concave and integrable, then so is their convolution $f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$.

(11) Let $f : \mathbb{R} \to \mathbb{R}_+$. Show that if $0 < p < q$, then then the property "$f^q(t)$ is concave on its support" implies "$f^p(t)$ is concave on its support". Passing to the limit $p \to 0$, show that this implies that $\log f(t) : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave (i.e. that $f$ is log-concave).

(12) Show that if $A$ is compact in $\mathbb{R}^n$ then so is its Steiner symmetrization $S_H A$.

(13) Let $A$ be a compact subset of $\mathbb{R}^n$. Show that $S_H A = A$ for all centered hyperplanes $H$, if and only if $A$ is a centered Euclidean ball.

(14) (a) Let $K \subset \mathbb{R}^n$ be a convex body, and let $E$ be an $m$-dimensional subspace. Show that the function:

$$E \ni y \mapsto Vol(K \cap (y + E^{\perp}))^{\frac{1}{n-m}} ,$$

is concave on its support.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a function so that $f^{1/k}$ is concave on its support, for some integer $k$. Denoting:

$$g(y) := \int_{y+E^{\perp}} f(x)dx ,$$

show that the function:

$$E \ni y \mapsto g(y)^{\frac{1}{n+k-m}} ,$$

is concave on its support. Taking $k \to \infty$, deduce that when $f$ is log-concave, then so is its marginal $g$. (Remark: compare with exercise 8(b)).

(15) Show that the isoperimetric inequality on $\mathbb{R}^n$:

$$|\partial A| \geq n \, |D_n|^{1/n} \, |A|^{(n-1)/n} ,$$

implies the Brunn-Minkowski inequality when one of the sets is a Euclidean ball $D$:

$$\mathrm{Vol}(A + D)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(D)^{1/n} ,$$

(say for sets $A$ with smooth boundary).

(16) Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$ and $E$ a linear subspace of dimension $k$. We proved in class the Rogers–Shephard (RS) inequality:

$$|K| \leq \left| K \cap E^{\perp} \right| |Proj_E K| \leq \binom{n}{k} |K| .$$

(a) Provide examples demonstrating the sharpness of each inequality.

(b) As a corollary of RS, deduce that if $K$ is not-necessarily origin-symmetric, then:

$$|K + (-K)| \leq \binom{2n}{n} |K| \ .$$

Hint: construct an appropriate $L \subset \mathbb{R}^{2n}$ and use RS with an appropriate $E$. Note that for the RHS of the RS inequality, no assumption on origin-symmetric is required.

(17) Prove that $\mathbb{R}_+ \ni r \mapsto Vol(K \cap rD_n)^{1/n}$ is a concave function, for any convex $K$ in $\mathbb{R}^n$.

(18) Recall the correspondence between origin-symmetric convex bodies $K$ and norms on $\mathbb{R}^n$, described in Exercise 1. Show that:
(a) $\|x\|_{K \cap L} = \max(\|x\|_K, \|y\|_L)$.
(b) $\|z\|_{conv(K \cup L)} = \inf\{\|x\|_K + \|y\|_L ; z = x + y\}$.
(c) $\|z\|_{K+L} = \inf\{\max(\|x\|_K, \|y\|_L); z = x + y\}$.

(19) Recall the definition of the dual norm: $\|y\|^* := \sup_{\|x\| \leq 1} |\langle x, y\rangle|$. Show that:
(a) $\|\cdot\|^*$ is indeed a norm.
(b) $(\|\cdot\|^*)^* = \|\cdot\|$. Hint: prove that if $x \notin K$ then there exists a separating hyperplane.
(c) $\|\cdot\|_1 \leq \|\cdot\|_2$ implies $\|\cdot\|_1^* \geq \|\cdot\|_2^*$.

(20) Let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$, and recall that the unit-ball of $\|\cdot\|_K^*$ is the polar-set $K^\circ$. Show that:
(a) $(K_1 \cap K_2)^\circ = conv(K_1^\circ \cup K_2^\circ)$.
(b) $h_{K_1+K_2} = h_{K_1} + h_{K_2}$, where $h_L = \|\cdot\|_L^*$ is the support function of $L$.
(c) If $K = [-a\theta, a\theta]$, $\theta \in S^{n-1}$, then $K^\circ = \{x \in \mathbb{R}^n \ ; \ |\langle x, \theta\rangle| \leq 1/a\}$.
(d) $K \subset L$ implies $L^\circ \subset K^\circ$.
(e) If $T \in GL(n)$ then $(TK)^\circ = T^{-*}(K^\circ)$, where $T^{-*} = (T^{-1})^*$. In particular, $(\lambda K)^\circ = \lambda^{-1}K^\circ$.
(f) Conclude that $d_{BM}(K, L) = d_{BM}(K^\circ, L^\circ)$.

(21) Show that $M^*(S_H K) \leq M^*(K)$, where $S_H K$ denotes the Steiner symmetral of $K$ with respect to an arbitrary linear hyperplane $H$. As a corollary, deduce Urysohn's inequality: $|K| = |D_n|$ implies $M^*(K) \geq M^*(D_n) = 1$.

(22) Let $X$ be a $G$-homogeneous space, where $G$ is a compact Hausdorff topological group. Recall that the Haar measure $\mu_X$ on $X$ is defined by: $\int_X f(x) d\mu_X(x) = \int_G f(gx_0) d\mu_G(g)$, for any fixed $x_0 \in X$. Show that $\mu_X$ is the unique (up to positive multiple) $G$-left-invariant measure on $X$.

(23) (a) Let $\theta_0 \in S^{n-1}$, and let $G_{n,k}$ denote the Grassmanian of all $k$-dimensional linear subspaces of $\mathbb{R}^n$. Show that the space $G_{n,k}^{\theta_0} := \{E \in G_{n,k} \ ; \ \theta_0 \in E\}$ is $O_{\theta_0}$-homogeneous for an appropriate compact group $O_{\theta_0}$.

(b) Using uniqueness of the Haar measure, show that for any continuous function $f$ on $G_{n,k}$:

$$\int_{S^{n-1}} \int_{G^\theta_{n,k}} f(E) d\mu_{G^\theta_{n,k}}(E) d\mu_{S^{n-1}}(\theta) = \int_{G_{n,k}} f(E) d\mu_{G_{n,k}}(E) \ ,$$

where $\mu_X$ denotes the Haar **probability** measure on the compact space $X$.

(c) Similarly, show that for any continuous function $f$ on $S^{n-1}$:

$$\int_{G_{n,k}} \int_{S(E^k)} f(\theta) d\mu_{S(E^k)}(\theta) d\mu_{G_{n,k}}(E^k) = \int_{S^{n-1}} f(\theta) d\mu_{S^{n-1}}(\theta) \ ,$$

where $E^k \in G_{n,k}$ and $S(E^k) := E^k \cap S^{n-1}$.

(24) Recall that Steiner's formula states that:

$$Vol(K + tD_n) = \sum_{i=0}^{n} \binom{n}{i} W_{n-i}(K) t^i \ .$$

Let $K$ be a smooth convex body. Prove using Steiner's formula and integration in polar coordinates that $W_1(K) = Vol(D_n) M^*(K)$, where recall:

$$M^*(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) \ .$$

Here $\sigma$ is the Haar probability measure on $S^{n-1}$ and $h_K$ is the support function of $K$.

(25) Prove the following generalized version of Kubota's formula:

$$(*) \qquad W_i(K) = \frac{Vol(D_n)}{Vol(D_q)} \int_{G_{n,q}} W_i(Proj_E K) d\mu_{G_{n,q}}(E) \quad \forall 0 \le i \le q \le n,$$

where $\mu_G$ denotes the Haar probability measure on the corresponding homogeneous space $G$. Guidance:

(a) Using $Vol((K+tD_n)+rD_n) = Vol(K+(t+r)D_n)$ and Steiner's formula (see previous exercise), deduce that:

$$W_j(K + rD_n) = \sum_{i=0}^{j} \binom{j}{i} W_{j-i}(K) r^i \ .$$

(b) Note that the case $q = i = n - 1$ in (*) is exactly Cauchy's formula, proved in class:

$$\frac{Vol_{n-1}(\partial K)}{n} = W_{n-1}(K) = C_n \int_{G_{n,n-1}} Vol_{n-1}(Proj_E K) d\mu_{G_{n,n-1}}(E).$$

(c) Express the polynomial $W_{n-1}(K + tD_n)$ in two different manners: on the one hand use (a), and on the other hand use (b) and apply Steiner's formula to $Vol_{n-1}(Proj_E(K+tD_n))$. Equating the coefficients of $t^{n-1-i}$

in both representations, deduce (*) for $q = n-1$ and general $0 \leq i \leq n-1$, i.e.:

$$(**) \qquad W_i(K) = \frac{Vol(D_n)}{Vol(D_{n-1})} \int_{G_{n,n-1}} W_i(Proj_E K) d\mu_{G_{n,n-1}}(E) \quad \forall 0 \leq i \leq n-1.$$

(d) Deduce (*) in full generality by reverse induction on $q$. Assume it holds for $q+1$, i.e.:

$$W_i(K) = C_{n,i} \int_{G_{n,q+1}} W_i(Proj_E K) d\mu_{G_{n,q+1}}(E).$$

Proceed by applying (**) with $n = q+1$ to the integrand, and using double integration:

$$\int_{G_{n,q+1}} \int_{G_{E,q}} f(F) d\mu_{G_{E,q}}(F) d\mu_{G_{n,q+1}(E)} = \int_{G_{n,q}} f(F) d\mu_{G_{n,q}}(F).$$

Here $G_{E,q}$ denotes the Grassmanian of all $q$ dimensional subspaces of $E \in G_{n,q+1}$.

(26) (a) Show that $M^*([0,x]) = c_n |x|$ with $c_n \simeq \frac{1}{\sqrt{n}}$.

(b) Given a parallelepiped with side-lengths $\{a_i\}_{i=1,\ldots,n}$ (an affine image of a cube $[0,1]^n$), denote its 1-dimensional perimeter by $\mathrm{per}_1(K) := \sum_{i=1}^n a_i$. Deduce from (a) that $M^*(K) = c_n \mathrm{per}_1(K)$.

(c) If $K_1, K_2$ are two parallelepipeds so that $\mathrm{per}_1(K_1) < \mathrm{per}_1(K_2)$, is it possible to place $K_2$ inside $K_1$ (i.e. find a Euclidean isometry $T$ so that $T(K_2) \subset K_1$)?

(27) Recall that the Alexandrov–Fenchel inequality states that:

$$V(K_1, K_2, L_3, \ldots, L_n)^2 \geq V(K_1, K_1, L_3, \ldots, L_n) V(K_2, K_2, L_3, \ldots, L_n) ,$$

where $V$ denotes mixed-volume. We use the notation $V(K; i, L; n-i)$ to denote the mixed volume of the tuple where $K$ appears $i$ times and $L$ appears $n-i$ times. Using only this, deduce:

(a) $V(K; i, L; n-i) \geq Vol(K)^{i/n} Vol(L)^{(n-i)/n}$.

(b) More generally, deduce Minkowski's inequality:

$$V(K_1, \ldots, K_n) \geq (\Pi_{i=1}^n Vol(K_i))^{1/n} .$$

(c) Specializing (a) to the case $i = n-1$, we have Minkowski's inequality:

$$V(K; n-1, L; 1) \geq Vol(K)^{(n-1)/n} Vol(L)^{1/n} .$$

Note and explain why the case of $L = D_n$ corresponds to the Euclidean isoperimetric inequality (for convex bodies).

(d) Use the inequality in (c) above and the multi-linearity of the mixed-volumes $V$ to deduce the Brunn–Minkowski inequality:

$$Vol(K+L)^{1/n} \geq Vol(K)^{1/n} + Vol(L)^{1/n} .$$

(e) Generalize all of the above to show that:

$$W_i(K+L)^{1/i} \geq W_i(K)^{1/i} + W_i(L)^{1/i} \ , \ \forall i = 1,\ldots,n \ ,$$

where recall $W_i(K) = V(K; i, D_n; n-i)$.

(28) Calculate the isotropic constant of the Euclidean ball $D_n$ and the cross-polytope $B_1^n$ having volume 1 in $\mathbb{R}^n$. Find their asymptotic values as $n \to \infty$.

(29) Complete the proof of John's theorem, stating that if $D_n \subset K$ is the maximal volume ellipsoid inside an origin-symmetric convex body $K$ in $\mathbb{R}^n$, then $K \subset \sqrt{n} D_n$. Do this by directly arguing that otherwise, there would exist a point $x_0 \notin K$ with $|x_0| > \sqrt{n}$, and hence by convexity $K \supset C := conv(D_n \cup \{\pm x_0\})$ ; a contradiction will follow if an ellipsoid is found inside $C$ having even larger volume than $D_n$.

(30) (a) Show that $K = \sqrt{n} B_1^n$ is in John's position (can use the characterization we learned in class via the isotropic measure supported on contact points of $\partial D_n$ and $\partial K$).
(b) Deduce by duality that the minimal-volume ellipsoid containing $B_\infty^n$ is $\sqrt{n} D_n$.
(c) Deduce from (a) and (b) that $d_{BM}(B_\infty^n, D_n) = \sqrt{n}$.

(31) (a) Calculate the volume of $B_p^n$, the unit-ball of $\ell_p^n$, by integrating the measure $\exp(-\|x\|_{\ell_p^n}^p)dx$ on $\mathbb{R}^n$.
(b) Deduce that when $p \in [1,2]$, the volume-ratio $\mathrm{vr}(B_p^n)$ of $B_p^n$ is uniformly bounded by a numeric constant $C > 0$, independent of $n$ and $p \in [1,2]$. Here the volume-ratio $\mathrm{vr}(K)$ of a body $K \subset \mathbb{R}^n$ is defined as:

$$\mathrm{vr}(K) := \min \left\{ \left( \frac{Vol(K)}{Vol(\mathcal{E})} \right)^{1/n} \ ; \ \text{ellipsoid } \mathcal{E} \subset K \right\} .$$

(c) If $\tilde{D}_n$ denotes the (centered) Euclidean ball in $\mathbb{R}^n$ of volume 1, calculate $Vol_{n-1}(\tilde{D}_n \cap H)$ for a (co-dimension 1) hyperplane $H$, and show that it tends to $\sqrt{e}$ as $n \to \infty$.

(32) Verify that if $\mu_{S^n}$ is the Haar probability measure on $S^n$, then $\mu_{S^n}(B_\theta(x_0)) \geq (c\theta)^n$ for all $\theta \in [0, \pi/2]$, where $B_\theta(x_0)$ is a geodesic ball of radius $\theta$ on $S^n$, $x_0 \in S^n$ is any point, and $c > 0$ is an appropriate constant.

(33) Assume that $K$ is an origin-symmetric convex body in $\mathbb{R}^n$ with smooth boundary. Show that:

$$\nabla \|x\|_K = \frac{1}{h_K(n_{\partial K}(x))} \cdot n_{\partial K}(x) \quad \forall x \in \partial K,$$

where $n_{\partial K}(x)$ denotes the unit outer normal to $\partial K$ at $x$, and $h_K$ is the corresponding support function.

(34) Consider the metric-measure space $(S^n, d, \mu_{S^n})$, where $\mu = \mu_{S^n}$ it the corresponding Haar probability measure, and $d$ is the geodesic distance on $S^n$. Let $f : S^n \to \mathbb{R}$ be a 1-Lipschitz function, let $m(f)$ denote its median, and define $E(f) = \int f d\mu$.
   (a) Show that $|E(f) - m(f)| \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
   (b) Show that $0 \leq \sqrt{E(f^2)} - E(|f|) \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
   (c) Deduce concentration around $E(f)$, and if $f \geq 0$, also around $\sqrt{E(f^2)}$. In other words, show that:
$$\mu(x \in S^n \; ; \; |f(x) - A_f| \geq r) \leq C \exp(-\frac{n-1}{2} r^2) \; ,$$
   where $A_f$ is either $E(f)$, and when $f \geq 0$, also $\sqrt{E(f^2)}$, for some constant $C > 0$.
   (d) Show that if $f$ is $L$-Lipschitz, then:
$$\mu(x \in S^n \; ; \; f(x) - m(f) \geq r) \leq \sqrt{\frac{\pi}{8}} \exp(-\frac{n-1}{2L^2} r^2) \; .$$
   (e) More generally, show that:
$$\mu(x \in S^n \; ; \; f(x) - m(f) \geq \omega_f(r)) \leq \sqrt{\frac{\pi}{8}} \exp(-\frac{n-1}{2} r^2) \; ,$$
   where $\omega_f(r) = \sup \{|f(x) - f(y)| \; ; \; d(x,y) \leq r\}$ denotes the modulus of continuity of $f$.

(35) Show that the Gaussian isoperimetric inequality on $(\mathbb{R}^n, |\cdot|, \gamma_n)$, where $\gamma_n$ denotes the standard Gaussian measure on $\mathbb{R}^n$, namely:
$$\gamma_n(A) = \gamma_n(H) \quad \Rightarrow \quad \gamma_n^+(A) \geq \gamma_n^+(H) \; ,$$
implies:
$$\gamma_n(A) = \gamma_n(H) \quad \Rightarrow \quad \gamma_n(A_r) \geq \gamma_n(H_r) \; , \; \forall r > 0 \; .$$
Here $H$ is (any) half-plane, $\gamma_n^+(C)$ denote the Gaussian boundary measure of a Borel set $C \subset \mathbb{R}^n$, and $C_r$ denote the $r$-extension of $C$.
Guidance: mimic the proof we saw in class for the Sphere.

(36) Let $A$ denote a $k \times n$ random matrix with i.i.d. standard Gaussian entries. Show that with very high-probability (quantify this!), the map $T : \mathbb{R}^n \to \mathbb{R}^k$ given by $T(x) := \frac{1}{\sqrt{k}} Ax$ is a good "Johnson-Lindenstrauss" map, i.e.:
$$(1-\varepsilon)|z|_{\mathbb{R}^n} \leq |T(z)|_{\mathbb{R}^k} \leq (1+\varepsilon)|z|_{\mathbb{R}^n}$$
with very high-probability for a fixed $z \in \mathbb{R}^n$.

(37) Assume that on a metric-measure (probability) space $(\Omega, d, \mu)$ we know that:
$$\forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A_r) < K(r) \;\; \forall r > 0 \; .$$
Show that:
$$\forall A \subset \Omega \quad \mu(A) \geq K(\varepsilon_0) \quad \Rightarrow \quad \mu(\Omega \setminus A_{r+\varepsilon_0}) < K(r) \;\; \forall r > 0 \; .$$

Deduce that on $S^n$, we have the following concentration property:

$$\mu(A) \geq \sqrt{\pi/8}\exp(-\frac{n-1}{2}\varepsilon_0^2) \quad \Rightarrow \quad \mu(S^n \setminus A_{2\varepsilon_0}) \leq \sqrt{\pi/8}\exp(-\frac{n-1}{2}\varepsilon_0^2) \ .$$

(38) Let $k = k(n-1, \varepsilon)$ be as defined in class, namely a value of $k$ which guarantees that for any continuous function on $S^{n-1}$:

$$\mu_{G_{n,k}}\left\{E^k \in G_{n,k} \ ; \ \forall x \in S(E^k) \ \ |f(x) - m(f)| \leq \omega_f(\varepsilon)\right\} \geq 1 - \exp(-\frac{n-2}{4}\varepsilon^2) \ .$$

Assume that $A \subset S^{n-1}$ is such that for every $E^k \in G_{n,k}$, $A \cap E^k \neq \emptyset$. Then there exists a $E_0^k \in G_{n,k}$ so that $E_0^k \cap S^{n-1} \subset A_{2\varepsilon}$.
Guidance: define the right Lipschitz function $f$ and use a similar argument to the one in the previous exercise.

(39) Recall that:

$$M_p(K) := \left(\int_{S^{n-1}} \|x\|_K^p \, d\mu_{S^{n-1}}(x)\right)^{1/p} \ , \ \Gamma_p(K) := \left(\int_{\mathbb{R}^n} \|x\|_K^p \, d\gamma_n(x)\right)^{1/p} \ ,$$

and that we showed in class that $\Gamma_2(K) = \sqrt{n}M_2(K)$ for any (say origin-symmetric) convex $K$.
  (a) Show by direct calculation that $\Gamma_1(K) \simeq \sqrt{(n)}M_1(K)$. Here $A \simeq B$ means that the ratio of $A$ and $B$ is bounded from above and from below by two numerical constants, independent of the dimension.
  (b) Show the "reverse Jensen inequality": $M_p(K) \leq C\sqrt{p}M_1(K)$.
  Guidance: we showed that:

$$\mu_{S^{n-1}}(|\|x\|_K - M_1(K)| \geq tM_1(K)) \leq C\exp(-cnt^2(M_1(K)/b(K))^2) \leq C\exp(-c't^2) \ ;$$

  now repeat the proof of the reverse Holder inequality we gave in class for semi-norms on log-concave distributions (corollary of Borell's Lemma).
  (c) Deduce (a) again from (b) without performing any calculations.

(40) Prove that for any ellipsoid $\mathcal{E}$ in $\mathbb{R}^k$ (i.e. a linear image of a Euclidean ball), there exists a subspace $F$ of dimension $\lceil k/2 \rceil$ so that $\mathcal{E} \cap F = R_{\mathcal{E}} B_F$, i.e. the section by $F$ is a Euclidean ball (of a certain radius $R_{\mathcal{E}}$).

(41) Prove the following estimates:

$$\frac{1}{\sqrt{2\pi}}\frac{t}{t^2+1}\exp(-t^2/2) \leq \text{Prob}(\gamma_1 > t) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{t}\exp(-t^2/2) \quad \forall t > 0 \ .$$

Hint for the left-hand inequality - try to take derivatives.

(42) Show that for any $E \in G_{n,k}$, $\Gamma_2(K) \geq \Gamma_2(K \cap E)$, and hence $M_2(K \cap E) \leq \sqrt{n/k}M_2(K)$. Using $k = 1$, deduce that $b(K) \leq \sqrt{n}M_2(K)$.
Guidance: repeat the proof of the proposition where we showed that in John's position, $M(K)/b(K) \geq c\sqrt{\log n}/\sqrt{n}$.

(43) Show that $\text{Med}(\|\cdot\|)\text{Med}(\|\cdot\|^*) \geq 1$. Here $\text{Med} = m$ denotes a median of its argument on $S^{n-1}$.

(44) Show that for any convex set $K$ and Euclidean ball $D$, $N(K,D) = \bar{N}(K,D)$, where $N$ is the covering number, and $\bar{N}$ is the covering number where the points are required to all lie inside $K$.

(45) Show that if $K, T, D$ are three origin-symmetric convex bodies, then for any $z \in \mathbb{R}^n$:
$$|(K \cap (z + D)) + T| \leq |(K \cap D) + T| \ .$$
Hint: use Brunn's concavity principle and the origin-symmetry.

(46) Given an origin-symmetric $K$, an $M$-ellipsoid $\mathcal{E}_K$ of $K$ was defined as an ellipsoid $\mathcal{E}$ so that:
$$\max\{N(K, \mathcal{E}_K), N(\mathcal{E}_K, K), N(K^\circ, \mathcal{E}_K^\circ), N(\mathcal{E}_K^\circ, K^\circ)\} \leq \exp(Cn),$$
for some constant $C > 0$. Show that it is always possible to ensure that $|\mathcal{E}_K| = |K|$ (perhaps after changing the value of the constant $C$).

(47) Let $K, T$ denote two origin-symmetric convex bodies in $\mathbb{R}^n$.
 (a) Show that the following statements are equivalent:
  (i) $\exists C_1 > 0$ so that $N(K,T) \leq \exp(C_1 n)$.
  (ii) $\exists C_2 > 0$ so that $|K \cap T| \geq \exp(-C_2 n)\,|K|$.
  (iii) $\exists C_3 > 0$ so that $|\mathrm{Conv}(K^\circ \cup T^\circ)| \leq \exp(C_3 n)\,|K^\circ|$.
 The equivalence is in the sense that $C_i$ depends solely on $C_j$.
 Guidance: recreate the proof we have seen in class for the directions $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ for $T = D_n$. Close the equivalence (by showing $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$, or by showing $(iii) \Rightarrow (i)$ directly).
 (b) Use Part (a) to show that $N(K,T) \leq \exp(C_1 n)$ and $|K| \geq \exp(-C_1 n)\,|T|$ together imply $N(K^\circ, T^\circ) \leq \exp(C_2 n)$.
 (c) Use Part (b) to show that the following statements are equivalent (without quoting the duality-of-entropy conjecture or theorem):
  (i) $\exists C_1 > 0$ so that $\max\{N(K,T), N(T,K)\} \leq \exp(C_1 n)$.
  (ii) $\exists C_2 > 0$ so that $\max\{N(K^\circ,T^\circ), N(T^\circ,K^\circ)\} \leq \exp(C_2 n)$
 Again, the equivalence is in the sense that $C_1$ depends solely on $C_2$, and vice versa.
 (d) Show that the following statements are equivalent:
  (i) $D_n$ is an $M$-ellipsoid of $K$ with constant $C_1 > 0$ ("$K$ is in $M$-position").
  (ii) $\exists C_2 > 0$ so that $|K \cap D_n| \geq \exp(-C_2 n)\,|D_n|$ and $|\mathrm{Conv}(K \cup D_n)| \leq \exp(C_2 n)\,|D_n|$.
  (iii) $\exists C_3 > 0$ so that $|K^\circ \cap D_n| \geq \exp(-C_3 n)\,|D_n|$ and $|\mathrm{Conv}(K^\circ \cup D_n)| \leq \exp(C_3 n)\,|D_n|$.
 Again, the equivalence is in the sense that $C_i$ depends solely on $C_j$.