The Globalization Theorem for the Curvature-Dimension Condition

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joint work with Fabio Cavalletti (SISSA)

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L^2 Optimal Transport - Introduction

- $(X, \mathrm{d}, \mathfrak{m})$ Polish space with finite Borel measure is called m.m.s.
- L^2-Wasserstein distance between $\mu_0, \mu_1 \in \mathcal{P}(X)$:

$$W_2(\mu_0, \mu_1) := \inf \left\{ \left(\int_{X \times X} \mathrm{d}^2(x,y)\, \pi(dx, dy) \right)^{\frac{1}{2}} ; \begin{array}{c} \pi \in \mathcal{P}(X \times X), \\ \pi_0 = \mu_0, \pi_1 = \mu_1 \end{array} \right\} ;$$

  W_2 weakly metrizes $\mathcal{P}_2(X)$, yielding Polish $(\mathcal{P}_2(X), W_2)$.
- (X, d) is geodesic space iff $(\mathcal{P}_2(X), W_2)$ is geodesic space.
- Any geodesic $[0,1] \ni t \mapsto \mu_t \in \mathcal{P}_2(W)$ can be lifted to an Optimal Dynamical Plan $\nu \in \mathcal{P}(\mathrm{Geo}(X))$, so that $(\mathrm{e}_t)_\sharp(\nu) = \mu_t$, where:

  (evaluation map) $\mathrm{e}_t : \mathrm{Geo}(X) \ni \gamma \mapsto \gamma_t \in X.$

  $\mathrm{OptGeo}(\mu_0, \mu_1)$ = all such ν's with $(\mathrm{e}_i)_\sharp(\nu) = \mu_i$ $(i = 0, 1)$.
- (X, d) is called non-branching if geodesics do not branch at an interior-point into two separate geodesics.
  $(X, \mathrm{d}, \mathfrak{m})$ is called essentially non-branching (e.n.b.) if for any $\mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X, \mathrm{d}, \mathfrak{m})$, any ν is concentrated on non-branching subset $G \subset \mathrm{Geo}(X)$ (e.g. mGH limits of manifolds w/ Ric $\geq K$).

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Lott–Sturm–Villani Curvature-Dimension Condition

Definition (Sturm, Lott-Villani '04)

$(X, \mathrm{d}, \mathfrak{m})$ satisfies $\mathrm{CD}(K, N)$, $K \in \mathbb{R}$, $N \in [1, \infty]$ if
$\forall \mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X, \mathrm{d}, \mathfrak{m})$, $\exists \nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ s.t. $\forall N' \geq N$, $\forall t \in (0,1)$:

$$\int \rho_t^{-\frac{1}{N'}} d\mu_t \geq \int \left(\tau_{K,N'}^{(1-t)}(\ell(\gamma)) \rho_0^{-\frac{1}{N'}}(\gamma_0) + \tau_{K,N'}^{(t)}(\ell(\gamma)) \rho_1^{-\frac{1}{N'}}(\gamma_1) \right) \nu(d\gamma),$$

where $\mu_t = \rho_t \mathfrak{m}$ ($\mu_t \ll \mathfrak{m}$ automatically since $\mathfrak{m}(X) < \infty$).

Definition: $(X, \mathrm{d}, \mathfrak{m})$ satisfies $\mathrm{CD}_{loc}(K, N)$

if $\forall o \in X$, $\exists o \in X_o \subset X$, $\forall \mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X, \mathrm{d}, \mathfrak{m})$, $\mathrm{supp}(\mu_i) \subset X_o$,
$\exists \nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ s.t. above holds $\forall N' \geq N$, $\forall t \in (0,1)$.

Theorem (Alt. Definition of $\mathrm{CD}(K, N)$ for e.n.b. m.m.s. (GRSCM))

$(X, \mathrm{d}, \mathfrak{m}) \in \mathrm{CD}(K, N)$ *iff* $\forall \mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X, \mathrm{d}, \mathfrak{m})$, $\exists \nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$
(ν unique and $\nu = S_\sharp(\mu_0)$ for $S : X \to \mathrm{Geo}(X)$), s.t. $\forall t \in (0,1)$:
$\rho_t^{-\frac{1}{N}}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\ell(\gamma)) \rho_0^{-\frac{1}{N}}(\gamma_0) + \tau_{K,N}^{(t)}(\ell(\gamma)) \rho_1^{-\frac{1}{N}}(\gamma_1)$ $\forall \nu$*-a.e.* $\gamma \in \mathrm{Geo}(X)$.

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${\rho_t}^{-\frac{1}{N}}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\ell(\gamma)) {\rho_0}^{-\frac{1}{N}}(\gamma_0) + \tau_{K,N}^{(t)}(\ell(\gamma)) {\rho_1}^{-\frac{1}{N}}(\gamma_1)$ $\forall \nu$-*a.e.* $\gamma \in \mathrm{Geo}(X)$.

Distortion Coefficients σ and τ

The $\mathrm{CD}(K,N)$ condition:

$$\rho_t^{-\frac{1}{N}}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\ell(\gamma))\rho_0^{-\frac{1}{N}}(\gamma_0) + \tau_{K,N}^{(t)}(\ell(\gamma))\rho_1^{-\frac{1}{N}}(\gamma_1)$$

entails "τ-concavity" of $J_\gamma^{\frac{1}{N}}(t)$, where $J_\gamma(t) = \frac{\rho_0(\gamma_0)}{\rho_t(\gamma_t)}$ is the "Jacobian" of the transport map $T_t : x \mapsto \mathrm{e}_t \circ S(x)$ from γ_0 to γ_t. We have:

$$\tau_{K,N}^{(t)}(\theta) := \sigma_{K,N-1}^{(t)}(\theta)^{1-\frac{1}{N}} t^{\frac{1}{N}},$$

where coefficients $\sigma(t) = \sigma_{K,N-1}^{(t)}(\theta)$ and t control volume distortion perpendicular and parallel to γ (respectively).

$$\sigma''(t) + \theta^2 \frac{K}{N-1}\sigma(t) = 0\,, \quad \begin{matrix} \sigma(0) = 0 \\ \sigma(1) = 1 \end{matrix} \quad \Rightarrow \; \sigma_{K,N-1}^{(t)}(\theta) := \frac{\sin(t\theta\sqrt{\frac{K}{N-1}})}{\sin(\theta\sqrt{\frac{K}{N-1}})},$$

in accordance with the smooth Riemannian setting (Jacobi equation).

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Examples of m.m.s.'s satisfying $\mathrm{CD}(K, N)$

Remark: $\mathrm{CD}(K, N) \Rightarrow (\mathrm{supp}(\mathfrak{m}), d)$ is geodesic if $N < \infty$.

- (M^n, g, Vol_g) geodesically-convex,

$$\mathrm{Ric}_g \geq K \Leftrightarrow \mathrm{CD}(K, n) .$$

- $(M^n, g, \rho\mathrm{Vol}_g)$ geodesically-convex,

$$\mathrm{Ric}_g - \mathrm{Hess}_g \log \rho - \frac{1}{N-n} \nabla_g \log \rho \otimes \nabla_g \log \rho \geq K \Leftrightarrow \mathrm{CD}(K, N) .$$

- Finsler manifolds satisfy $\mathrm{CD}(0, n)$.
- Alexandrov spaces satisfy $\mathrm{CD}(0, n)$.
- Stable under mGH limits.
- $\mathrm{CD}(K, N)$ implies numerous geometric and analytic inequalities as in smooth setting.

Bakry–Émery, Cordero-Erausquin–McCann–Schmuckenshläger, Otto–Villani, von-Renesse–Sturm, Ohta, Petrunin, Lott–Sturm–Villani, Cavalletti–Mondino.

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Local-to-Global Question

Globalization Question (Sturm, Villani)

Let $(X, \mathrm{d}, \mathfrak{m})$ and assume $(\mathrm{supp}(\mathfrak{m}), \mathrm{d})$ is geodesic (or length space). Does $\mathrm{CD}_{loc}(K, N) \Rightarrow \mathrm{CD}(K, N)$? (as in the smooth setting)

Yes for non-branching spaces if $N = \infty$ (Sturm) or $K = 0$ (Villani).

No in general (Rajala): $\exists$ heavily branching $\mathrm{CD}_{loc}(0, 4)$ space which is not $\mathrm{CD}(K, N)$ for any $K \in \mathbb{R}$ and $N \in [1, \infty]$.

So restriction to non-branching, or more generally, e.n.b., is natural.

Main Result (Cavalletti–M. '16)

Yes for all $K \in \mathbb{R}$ and $N \in (1, \infty)$ if $\mathfrak{m}(X) < \infty$ and $(X, \mathrm{d}, \mathfrak{m})$ is e.n.b.

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The Challenge

Given a fixed W^2-geodesic $t \mapsto (\mathrm{e}_t)_\sharp(\nu)$, $\mathrm{CD}_{loc}(K,N)$ implies for ν-a.e. $\gamma \in \mathrm{Geo}(X)$ (setting as usual $J_\gamma(t) = \frac{1}{\rho_t(\gamma_t)}$):

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for all $[\alpha_0, \alpha_1] \subset [0,1]$ with $\alpha_1 - \alpha_0$ sufficiently small.

Previously known cases $\frac{K}{N} = 0 \Rightarrow \tau_{K,N}^{(t)} = t$ linear distortion, and so local t-concavity implies global t-concavity for $[\alpha_0, \alpha_1] = [0,1]$.

However, when $\frac{K}{N} \neq 0$, Deng–Sturm constructed a counterexample to local-to-global property of $\tau_{K,N}^{(t)}$-concavity.

Moral: the local-to-global property for $\frac{K}{N} \neq 0$, if true, cannot be obtained by a one-dimensional bootstrap argument on a *single* W_2-geodesic as above, and must follow from a stronger reason involving a *family* of W_2-geodesics *simultaneously*.

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The Challenge

Deng–Sturm: local-to-global for $\tau_{K,N}^{(t)}(\theta)$-concavity is false for $\frac{K}{N} \neq 0$.

However, Bacher–Sturm:

- Defined $\mathrm{CD}^*(K,N)$ by replacing $\tau_{K,N}^{(t)}(\theta)$ by weaker $\sigma_{K,N}^{(t)}(\theta)$:

$$J_\gamma^{\frac{1}{N}}((1-t)\alpha_0 + t\alpha_1) \geq \sigma_{K,N}^{(t)}(|\alpha_1 - \alpha_0|\,\theta) J_\gamma^{\frac{1}{N}}(\alpha_1) + \sigma_{K,N}^{(1-t)}(|\alpha_1 - \alpha_0|\,\theta) J_\gamma^{\frac{1}{N}}(\alpha_0) \quad \forall t \in [0,1].$$

  Now local-to-global for $\sigma_{K,N}^{(t)}(\theta)$-concavity is always true since:

$$\sigma''(t) + \theta^2 \frac{K}{N} \sigma(t) = 0 \Rightarrow (J_\gamma^{\frac{1}{N}})'' + \theta^2 \frac{K}{N} J_\gamma^{\frac{1}{N}} \leq 0 \text{ on } [\alpha_0, \alpha_1].$$

- For non-branching spaces, established local-to-global property: $\mathrm{CD}^*(K,N) \Leftrightarrow \mathrm{CD}^*_{loc}(K,N) \Leftrightarrow \mathrm{CD}_{loc}(K-\varepsilon,N) \;\; \forall \varepsilon > 0$.

Local-to-global challenge for $\mathrm{CD}(K,N)$: Disentangle σ ($\perp$) and t ($||$) contributions to Jacobian before integrating as above.

We will show: $J_\gamma(t) = L_\gamma(t) Y_\gamma(t)$, L_γ concave, $Y_\gamma^{\frac{1}{N-1}}$ $\sigma_{K,N-1}^{(t)}$-concave.

Then $\tau_{K,N}^{(t)}$-concavity of $J_\gamma^{\frac{1}{N}}$ follows by application of Hölder's inq.

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Then $\tau^{(t)}_{K,N}$-concavity of $J_\gamma^{\frac{1}{N}}$ follows by application of Hölder's inq.

The Challenge

Deng–Sturm: local-to-global for $\tau_{K,N}^{(t)}(\theta)$-concavity is false for $\frac{K}{N} \neq 0$.

However, Bacher–Sturm:

- Defined $\mathrm{CD}^*(K,N)$ by replacing $\tau_{K,N}^{(t)}(\theta)$ by weaker $\sigma_{K,N}^{(t)}(\theta)$:

$$J_\gamma^{\frac{1}{N}}((1-t)\alpha_0 + t\alpha_1) \geq \sigma_{K,N}^{(t)}(|\alpha_1 - \alpha_0|\,\theta) J_\gamma^{\frac{1}{N}}(\alpha_1) + \sigma_{K,N}^{(1-t)}(|\alpha_1 - \alpha_0|\,\theta) J_\gamma^{\frac{1}{N}}(\alpha_0) \quad \forall t \in [0,1].$$

Now local-to-global for $\sigma_{K,N}^{(t)}(\theta)$-concavity is always true since:

$$\sigma''(t) + \theta^2 \frac{K}{N}\sigma(t) = 0 \Rightarrow (J_\gamma^{\frac{1}{N}})'' + \theta^2 \frac{K}{N} J_\gamma^{\frac{1}{N}} \leq 0 \text{ on } [\alpha_0, \alpha_1].$$

- For non-branching spaces, established local-to-global property: $\mathrm{CD}^*(K,N) \Leftrightarrow \mathrm{CD}^*_{loc}(K,N) \Leftrightarrow \mathrm{CD}_{loc}(K-\varepsilon,N) \;\; \forall \varepsilon > 0$.

Local-to-global challenge for $\mathrm{CD}(K,N)$: Disentangle σ ($\perp$) and t ($||$) contributions to Jacobian before integrating as above.

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Then $\tau_{K,N}^{(t)}$-concavity of $J_\gamma^{\frac{1}{N}}$ follows by application of Hölder's inq.

L^1 Optimal-Transport and $\mathrm{CD}^1(K,N)$

L^1-Wasserstein distance, Monge–Kantorovich–Rubinstein duality:

$$W_1(\mu_0,\mu_1) = \inf_{\pi} \int_{X\times X} \mathrm{d}(x,y)\,\pi(dx,dy) = \sup_{u \text{ 1-Lipschitz}} \int_X u(d\mu_0 - d\mu_1)$$

Fix a 1-Lipschitz $u : (X,\mathrm{d}) \to \mathbb{R}$. Assume for simplicity $\mathrm{supp}(\mathfrak{m}) = X$.

- $R \subset X$ is called a transport-ray for u if $R = Im(\gamma)$, γ closed geodesic ($\ell(\gamma) \in (0,\infty]$), $|u(\gamma_t) - u(\gamma_s)| = \mathrm{d}(\gamma_t,\gamma_s)$, and R is maximal w.r.t. inclusion.
- $(X,\mathrm{d},\mathfrak{m})$ satisfies $\mathrm{CD}^1_u(K,N)$ if $\exists \{X_\alpha\}_{\alpha\in Q} \subset X$ s.t.:
  - $\mathfrak{m}\llcorner_{\mathcal{T}_u} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha)$, with $\mathfrak{m}_\alpha(X_\alpha) = 1$, for $\mathfrak{q}$-a.e. $\alpha \in Q$, where $\mathcal{T}_u = \{x \in X \,;\, \exists y \neq x \;\, |u(x) - u(y)| = \mathrm{d}(x,y)\}$.
  - For $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathrm{supp}(\mathfrak{m}_\alpha) = X_\alpha$.
  - For $\mathfrak{q}$-a.e. $\alpha \in Q$, X_α is a transport-ray for u.
  - For $\mathfrak{q}$-a.e. $\alpha \in Q$, one-dimensional $(X_\alpha,\mathrm{d},\mathfrak{m}_\alpha) \in \mathrm{CD}(K,N)$ ("$\mathrm{CD}(K,N)$ density Y_α", i.e. $Y_\alpha^{\frac{1}{N-1}}$ is $\sigma_{K,N-1}$-concave).
- $(X,\mathrm{d},\mathfrak{m})$ satisfies $\mathrm{CD}^1_{Lip}(K,N)$ if $\mathrm{CD}^1_u(K,N)$ $\forall$ 1-Lipschitz u.

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$\mathsf{CD}(K,N) \Rightarrow \mathsf{CD}^1_{Lip}(K,N)$

L^1-OT studied by Evans–Gangbo, Feldman–McCann, Ambrosio, etc..., but the relation between CD and the new CD^1_{Lip} is recent.

Key milestones, modulo new features in red:

- Heintze–Karcher '78: on (M^n, g, Vol_g), $\mathsf{CD}(K,n) \Rightarrow \mathsf{CD}^1_u(K,n)$ for all $u = d(\cdot, H)$, H is smooth oriented hypersurface.
- Generalized Heintze–Karcher (Bayle '04, Morgan '05): on $(M^n, g, \rho\mathrm{Vol}_g)$, $\mathsf{CD}(K,N) \Rightarrow \mathsf{CD}^1_u(K,N)$.
- Klartag '14: on $(M^n, g, \mathfrak{m} = \rho\mathrm{Vol}_g)$, $\mathsf{CD}(K,N) \Rightarrow \mathsf{CD}^1_{Lip}(K,N)$; No smoothness assumed on 1-Lipschitz u!
- Given $\int f \, d\mathfrak{m} = 0$, Klartag applied this to maximizing u in $W_1(f_+\mathfrak{m}, f_-\mathfrak{m})$, obtaining a 1-D "localization" with $\int_{X_\alpha} f d\mathfrak{m}_\alpha = 0$; previously known for $M^n = \mathbb{R}^n$ using bisection method of Payne–Weinberger, Gromov–Milman, Kannan–Lovász–Simonovits.
- Cavalletti–Mondino '15: on e.n.b. $(X, \mathsf{d}, \mathfrak{m})$, $\mathfrak{m}(X) < \infty$, $N < \infty$, geodesic, $\mathsf{CD}_{loc}(K,N) \Rightarrow \mathsf{CD}^1_{Lip}(K,N)$. Remark: CD_{loc} is enough since in 1-D, $\mathsf{CD}_{loc}(K,N) \Rightarrow \mathsf{CD}(K,N)$ easily.

Our plan: $\mathrm{CD}_{loc}(K,N)$ + geodesic $\Rightarrow_{\text{Cav–Mon}} \mathrm{CD}^1_{Lip}(K,N) \Rightarrow_{??} \mathrm{CD}(K,N)$.

Theorem (Cavalletti–M. '16)

$(X, \mathrm{d}, \mathfrak{m})$ *e.n.b.*, $\mathfrak{m}(X) < \infty$, $K \in \mathbb{R}$, $N \in (1,\infty)$. *TFAE:*

- $\mathrm{CD}(K,N)$.
- $\mathrm{CD}^1_{Lip}(K,N)$.
- $\mathrm{CD}^1(K,N)$ *(only need* $u(x) = \mathrm{sgn}(f(x))\mathrm{dist}(x,\{f=0\})$*).*
- $\mathrm{CD}^*(K,N)$ *(Bacher–Sturm, recall* $\mathrm{CD}^*_{loc}(K,N) \Leftrightarrow \mathrm{CD}_{loc}(K-,N)$*).*
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If in addition $(\mathrm{supp}(\mathfrak{m}), \mathrm{d})$ *is length-space, these are equivalent to:*

- $\mathrm{CD}_{loc}(K,N)$.

Starting point for showing $\mathrm{CD}^1 \Rightarrow \mathrm{CD}$:

- $\mathrm{CD}^1_{\mathrm{d}(\cdot,o)}(K,N) \Rightarrow \mathrm{MCP}(K,N)$ ($o \in X_\alpha$ by maximality) $\Rightarrow$ proper.
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Proof - Preliminaries

$$W_2^2(\mu_0,\mu_1) = \inf_{\pi} \int_{X\times X} \mathrm{d}^2(x,y)\,\pi(dx,dy) = \sup_{\varphi} \int_X \varphi(x)\mu_0(dx) + \int_X \varphi^c(y)\mu_1(dy),$$

where $\varphi^c(y) = \inf_{x\in X} \frac{\mathrm{d}(x,y)^2}{2} - \varphi(x)$ is the Kantorovich dual.
$W_2(\mu_0,\mu_1) < \infty \Rightarrow$ sup attained on $\varphi = (\varphi^c)^c$, "Kantorovich potential".

Any $\nu \in \mathrm{OptGeo}(\mu_0,\mu_1)$ concentrated on "Kantorovich geodesics":

$$G_\varphi = \left\{\gamma \in Geo(X)\ ;\ \varphi(\gamma_0) + \varphi^c(\gamma_1) = \ell(\gamma)^2/2\right\}.$$

Hopf–Lax semi-group: $Q_t f(y) = \inf_{x\in X} \frac{\mathrm{d}(x,y)^2}{2t} + f(x)$.
Interpolating potentials: $\varphi_0 = \varphi$, $\varphi_1 = -\varphi^c$, $-\varphi_t = Q_t(-\varphi)$, $t \in [0,1]$.
$\Rightarrow (t-s)\varphi_s$ is Kantorovich potential for (μ_s,μ_t).

Formally: $\mu_1 = (\exp(-\nabla\varphi))_\sharp \mu_0$; $\mu_t = (\exp(-(t-s)\nabla\varphi_s))_\sharp \mu_s$.
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1st Ingredient - Change-of-Variables Formula

Easy to check: $\varphi_s(\gamma_s) - \varphi_t(\gamma_t) = (t-s)\frac{\ell(\gamma)^2}{2} \;\; \forall t,s \in [0,1]$.

$\Rightarrow$ Define $\Phi_s^t := (\mathrm{e}_t \circ \mathrm{e}_s^{-1})_\sharp \varphi_s = \varphi_s \circ \mathrm{e}_s \circ \mathrm{e}_t^{-1} = \varphi_t + (t-s)\frac{\ell_t^2}{2}$.

Theorem (Change-of-Variables Formula)

Let $(X, \mathsf{d}, \mathfrak{m}) \in \mathsf{CD}^1(K,N)$, e.n.b., $\mathfrak{m}(X) < \infty$, $K \in \mathbb{R}$, $N \in (1,\infty)$. Then $\forall \mu_0, \mu_1 \in \mathcal{P}_2^{ac}(X, \mathsf{d}, \mathfrak{m})$, $\exists$ versions of $\rho_t = \frac{d\mu_t}{d\mathfrak{m}}$, such that for ν-a.e. $\gamma \in G_\varphi^+$, for a.e. $t,s \in (0,1)$, $\exists \partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t) > 0$, and:

$$\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \frac{\ell(\gamma)^2}{\partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t)} h_s(t) \quad \text{for a.e. } t,s \in (0,1),$$

with $([0,1], |\cdot|, h_s(t)dt) \in \mathsf{CD}(\ell(\gamma)^2 K, N)$, $h_s(s) = 1$.

- $\gamma \subset$ transport-ray for $u_s = \mathrm{sgn}(\varphi_s - \varphi_s(\gamma_s))\, \mathsf{d}(\cdot, \{\varphi_s = \varphi_s(\gamma_s)\})$, and $h_s(t)$ is obtained from $\mathsf{CD}^1_{u_s}(K,N)$ (cf. maximality of ray).
- No Lip regularity of Φ_s^t ($t \neq s$) available, so no co-area allowed.
- Used to prove $\mathsf{CD}^1 \Rightarrow \mathsf{CD}$ but new even under CD.

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- $\gamma \subset$ transport-ray for $u_s = \mathrm{sgn}(\varphi_s - \varphi_s(\gamma_s))\, \mathsf{d}(\cdot, \{\varphi_s = \varphi_s(\gamma_s)\})$, and $h_s(t)$ is obtained from $\mathsf{CD}^1_{u_s}(K,N)$ (cf. maximality of ray).
- No Lip regularity of Φ_s^t ($t \neq s$) available, so no co-area allowed.
- Used to prove $\mathsf{CD}^1 \Rightarrow \mathsf{CD}$ but new even under CD.

Change-of-Variables in the smooth setting

Fix γ and recall $u_s = \mathrm{sgn}(\varphi_s - \varphi_s(\gamma_s))\, \mathrm{d}(\cdot, \{\varphi_s = \varphi_s(\gamma_s)\})$.

$$\varphi_s(\gamma_s) - \varphi_t(\gamma_t) = (t-s)\frac{\ell(\gamma)^2}{2}.$$

- Let $T_t^s(x) := \exp_x(-(t-s)\nabla\varphi_s(x))$ be the L^2-OT map. $(T_t^s)_\sharp(\mu_s) = \mu_t$, so $\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \mathrm{Jac}|_{x=\gamma_s} T_t^s(x)$.
- Let $R_t^s(x) := \exp_x(-(t-s)\ell(\gamma)\nabla u_s)$ be the normal-ray map. We have $R_t^s(\gamma_s) = T_t^s(\gamma_s) = \gamma_t$. $\mathrm{CD}^1_{u_s}(K,N) \Rightarrow t \mapsto \mathrm{Jac}|_{x=\gamma_s} R_t^s(x)$ is $\mathrm{CD}(K\ell(\gamma)^2, N)$ density.
- Hence at γ_s: $\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \mathrm{Jac}\, T_t^s = \frac{\mathrm{Jac}\, T_t^s}{\mathrm{Jac}\, R_t^s}\mathrm{Jac}\, R_t^s =: \lambda_s(t) h_s(t)$.
- Calculating, $\lambda_s(t)$ depends on angle between the levels sets of Φ_s^t and φ_t at γ_t:

$$\frac{1}{\lambda_s(t)} = \frac{\left\langle \nabla\Phi_s^t(\gamma_t), \nabla\varphi_t(\gamma_t)\right\rangle}{\ell(\gamma)^2} = \frac{-\left\langle \nabla\Phi_s^t(\gamma_t), \gamma'(t)\right\rangle}{\ell(\gamma)^2} = \frac{\partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t)}{\ell(\gamma)^2},$$

where last equality follows since $\Phi_s^t(\gamma_t) = \varphi_s(\gamma_s)$ is constant in t.

Change-of-Variables in mm-setting

Tools: Fubini, Disintegration of measure, uniqueness of disintegration.

Given good $G \subset G_\varphi^+$, fix s and let $G_{a_s} := \{\gamma \in G \, ; \, \varphi_s(\gamma_s) = a_s\}$.

- As $\mathrm{e}_{[0,1]}(G_{a_s}) \subset \mathcal{T}_{u_s}$, disintegrate on transport-rays of u_s using $\mathrm{CD}^1_{u_s}$:

$$\mathfrak{m}\llcorner_{\mathrm{e}_{(0,1)}(G_{a_s})} = \int_{\mathrm{e}_s(G_{a_s})} (\mathrm{e}_s^{-1}(\beta))_\sharp (h_\beta^{a_s} \mathcal{L}^1 \llcorner_{(0,1)}) \mathfrak{q}^{a_s}(d\beta) = \int_{(0,1)} \mathfrak{m}_t^{a_s} \mathcal{L}^1(dt),$$

obtaining a new disintegration over the partition $\{\mathrm{e}_t(G_{a_s})\}_{t \in (0,1)}$.

Note that $\mathfrak{m}_t^{a_s} = (\mathrm{e}_t \circ \mathrm{e}_s^{-1})_\sharp (h_\cdot^{a_s}(t) \mathfrak{m}_s^{a_s})$.

- Disintegrate on partition $\{\mathrm{e}_t(G_{a_s})\}_{a_s \in \mathbb{R}}$:

$$\mathfrak{m}\llcorner_{\mathrm{e}_t(G)} = \int_{\varphi_s(\mathrm{e}_s(G))} \hat{\mathfrak{m}}_{a_s}^t \mathfrak{q}_s^t(da_s) =_{\mathfrak{q}_s^t \ll \mathcal{L}^1} \int_{\varphi_s(\mathrm{e}_s(G))} \mathfrak{m}_{a_s}^t \mathcal{L}^1(da_s).$$

Multiplying both sides by ρ_t, the LHS is $\mu_t = (\mathrm{e}_t)_\sharp(\nu)$, a W_2-geodesic. Therefore, same holds true for the conditional measures: for a.e. a_s, $\rho_t \mathfrak{m}_{a_s}^t = (\mathrm{e}_t)_\sharp(\nu_{a_s})$ is W_2-geodesic compatible with G ($\mathrm{supp}(\nu_{a_s}) \subset G_{a_s}$). Hence: $\rho_t \mathfrak{m}_{a_s}^t = (\mathrm{e}_t \circ \mathrm{e}_s^{-1})_\sharp (\rho_s \mathfrak{m}_{a_s}^s)$.

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$$\mathfrak{m}\llcorner_{e_{(0,1)}(G_{a_s})} = \int_{(0,1)} \mathfrak{m}_t^{a_s} \mathcal{L}^1(dt) \; , \; \mathfrak{m}\llcorner_{e_t(G)} = \int_{\varphi_s(e_s(G))} \mathfrak{m}_{a_s}^t \mathcal{L}^1(da_s).$$

$$\mathfrak{m}_t^{a_s} = (e_t \circ e_s^{-1})_\sharp (h_\cdot^{a_s}(t) \mathfrak{m}_s^{a_s}) \; , \; \rho_t \mathfrak{m}_{a_s}^t = (e_t \circ e_s^{-1})_\sharp (\rho_s \mathfrak{m}_{a_s}^s).$$

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Thm: for a.e. $s, t \in (0,1)$, $a_s \in \varphi_s(G_{a_s})$, $\mathfrak{m}_t^{a_s} = \partial_t \Phi_s^t \; \mathfrak{m}_{a_s}^t$.

Cor: Calculating Radon-Nykodim derivative:

$$\frac{\partial_\tau|_{\tau=t} \Phi_s^\tau(\gamma_t)}{\rho_t(\gamma_t)} = \left. \frac{\mathfrak{m}_t^{a_s}}{\rho_t \mathfrak{m}_{a_s}^t} \right|_{\gamma_t} = \left. \frac{h_\cdot^{a_s}(t) \mathfrak{m}_s^{a_s}}{\rho_s \mathfrak{m}_{a_s}^s} \right|_{\gamma_s} = \frac{h_s(t)}{\rho_s(\gamma_s)} \partial_\tau|_{\tau=s} \Phi_s^\tau(\gamma_s) = \frac{h_s(t)}{\rho_s(\gamma_s)} \ell(\gamma)^2$$

Formal Proof of Thm: write $\Phi_s^t(x) = \Phi_s(t, x)$.

$$e_t(G_{a_s}) = e_t(G) \cap \{x \; ; \; \Phi_s(t,x) = a_s\} = e_t(G) \cap \left\{x \; ; \; \Phi_s(\cdot, x)^{-1}(a_s) = t\right\}.$$

By formal coarea $\frac{\mathfrak{m}_t^{a_s}}{\mathfrak{m}_{a_s}^t} = \frac{|\nabla_x \Phi_s(t,x)|}{|\nabla_x \Phi_s(\cdot,x)^{-1}(a_s)|} = |-\partial_t \Phi_s(t,x)|$, since by implicit function thm: $\Phi(\Phi^{-1}(a,x),x) = a \Rightarrow \nabla_x \Phi + \partial_t \Phi \nabla_x \Phi^{-1} = 0$.

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$$\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \frac{\ell(\gamma)^2}{\partial_\tau|_{\tau=t}\Phi_s^\tau(\gamma_t)} h_s(t) \qquad \text{for a.e. } t,s \in (0,1).$$

Formally: $\Phi_s^t = \varphi_t + (t-s)\frac{\ell_t^2}{2}$, $\partial_t \varphi_t = \frac{1}{2}\ell_t^2$, $\partial_t \Phi_s^t = \ell_t^2 + (t-s)\partial_t \frac{\ell_t^2}{2}$.

Want: $\frac{1}{\rho_t(\gamma_t)} = L_\gamma(t) Y_\gamma(t)$, L_γ concave and $Y_\gamma^{\frac{1}{N-1}}$ $\sigma_{K,N-1}^{(t)}$-concave.

Main difficulty: need ∂_t of denominator, i.e. $\partial_t^2 \ell_t^2$, i.e. $\partial_t^3 \varphi_t$.

Theorem (On a general proper geodesic (X, d))

For any $\gamma \in G_\varphi$, if $\exists \frac{1}{\ell(\gamma)^2}\partial_\tau|_{\tau=t}\ell_\tau^2/2(\gamma_t)$ for a.e. $t \in (0,1)$ and coincides w/ absolutely continuous z, then $z'(t) \geq z(t)^2$ for a.e. $t \in (0,1)$.

The conclusion is equivalent to the assertion that:

$$(0,1) \ni r \mapsto L(r) := \exp\left(-\frac{1}{\ell(\gamma)^2}\int_{r_0}^r \partial_\tau|_{\tau=t}\frac{\ell_\tau^2}{2}(\gamma_t)dt\right) \text{ is concave },$$

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Formal argument in smooth Riemannian setting

Recall H-J: $\partial_t \varphi_t = \frac{1}{2}\ell_t^2 = \frac{1}{2}|\nabla \varphi_t|^2$, $\bar{z}(t) = \partial_t^2 \varphi_t(\gamma(t))$, $z(t) = \frac{\bar{z}(t)}{\ell(\gamma)^2}$.

(we evaluate all subsequent functions at $x = \gamma_t$). Calculate:

$$\bar{z}'(t) = \partial_t^3 \varphi_t + \left\langle \nabla \partial_t^2 \varphi_t, \gamma'(t) \right\rangle = \partial_t^3 \varphi_t - \left\langle \nabla \partial_t^2 \varphi_t, \nabla \varphi_t \right\rangle .$$

But taking two time derivatives in (H-J), we know that:

$$\partial_t^3 \varphi_t = \left\langle \nabla \partial_t^2 \varphi_t, \nabla \varphi_t \right\rangle + \langle \nabla \partial_t \varphi_t, \nabla \partial_t \varphi_t \rangle \quad \Rightarrow \quad \bar{z}'(t) = |\nabla \partial_t \varphi_t|^2 .$$

It follows by Cauchy–Schwarz that:

$$\bar{z}'(t) \geq \frac{\langle \nabla \partial_t \varphi_t, \nabla \varphi_t \rangle^2}{|\nabla \varphi_t|^2} = \frac{\langle \nabla \partial_t \varphi_t, \nabla \varphi_t \rangle^2}{\ell^2(\gamma)} = \frac{\bar{z}(t)^2}{\ell^2(\gamma)},$$

where last identity since $\partial_t \varphi_t(\gamma_t) = \ell_t^2/2(\gamma_t) = \ell(\gamma)^2/2$ is constant:

$$0 = \partial_t^2 \varphi_t + \langle \nabla \partial_t \varphi_t, \gamma'(t) \rangle = \bar{z}(t) - \langle \nabla \partial_t \varphi_t, \nabla \varphi_t \rangle .$$

$\bar{z}'(t) \geq \bar{z}(t)^2/\ell(\gamma)^2$ - In reality...

Previous argument (wrongly) suggests that Hilbertianity is crucial.

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3rd Ingredient - Rigidity of CoV Formula

For ν-a.e. $\gamma \in G_\varphi^+$, the Change-of-Variables Formula yields:

$$\frac{\rho_s(\gamma_s)}{\rho_t(\gamma_t)} = \frac{h_s(t)}{1 + (t - s)\frac{\partial_\tau|_{\tau=t}\ell_\tau^2/2(\gamma_t)}{\ell^2(\gamma)}} \quad \text{for a.e. } t, s \in (0,1).$$

Note separation of variables on LHS and linearity in s in denominator; this allows to gain additional order of regularity in t, s. Indeed:

$t \mapsto \rho_t(\gamma_t), h_s(t)$ are locally Lipschitz, hence $\frac{\partial_\tau|_{\tau=t}\ell_\tau^2/2(\gamma_t)}{\ell^2(\gamma)} = z(t)$ a.e. with z locally Lipschitz, and hence $z' \geq z^2$ a.e. by 2nd Ingredient. Moreover, we can redefine $\{h_s\}_{s \in S}$ so that $s \mapsto h_s(t)$ is loc. Lipschitz.

Theorem

Assume that on $(0,1)$, $\rho(t)$ locally Lipschitz, $\{h_s(t)\}_{s\in(0,1)}$ are $\mathrm{CD}(K_0, N)$ densities, $z'(t) \geq z^2(t)$ a.e., and:

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Formal Argument using rigidity

Fix any $r_0 \in (0,1)$, and define the functions L and Y by:

$$\log L(r) := -\int_{r_0}^{r} z(s)ds \; , \; \log Y(r) := \int_{r_0}^{r} \partial_t|_{t=s} \log h_s(t) ds.$$

$$\Longrightarrow \; \log \frac{\rho(r_0)}{\rho(r)} = \int_{r_0}^{r} \partial_t|_{t=s} \log \frac{\rho(s)}{\rho(t)} ds = \int_{r_0}^{r} \partial_t|_{t=s} \log h_s(t) ds$$

$$- \int_{r_0}^{r} \partial_t|_{t=s} \log(1 + (t-s)z(t)) ds = \log Y(r) + \log L(r).$$

We saw that $z'(t) \geq z(t)^2$ yields concavity of L. For all $r \in (0,1)$:

$$(\log Y)'(r) = \partial_t|_{t=r} \log h_r(t),$$

$$(\log Y)''(r) = \partial_t^2|_{t=r} \log h_r(t) + \partial_s \partial_t|_{t=s=r} \log h_s(t).$$

$$\partial_s \partial_t|_{t=s=r} \log h_s(t) =_{Rigidity} \partial_s \partial_t|_{t=s=r} \log(1+(t-s)z(t)) = -z'(r) + z^2(r) \leq 0.$$

Hence, using differential char. of $\mathrm{CD}(K_0, N)$ density for $h_r(t)$ at $t = r$:

$$(\log Y)''(r) + \frac{((\log Y)'(r))^2}{N-1} \leq \partial_t^2|_{t=r} \log h_r(t) + \frac{(\partial_t|_{t=r} \log h_r(t))^2}{N-1} \leq -K_0 \qquad \square$$

Thank you very much!