ISOPERIMETRIC INEQUALITIES, CONCENTRATION OF MEASURE, CONVEXITY AND APPLICATIONS

(note that the statement of one exercise may be used for the proof of another).

(1) Lower semi-continuity with respect to volume convergence:

  (a) Show that if $\{A_i\}, A$ are open sets with smooth boundaries in $\mathbb{R}^n$, and $\mathrm{Vol}(A_i \triangle A) \to 0$, then $H^{n-1}(\partial A) \leq \liminf_{i\to\infty} H^{n-1}(\partial A_i)$.
  (b) Show that the same statement as above is true for general sets of finite De-Giorgi perimeter, if one replaces in (a) every occurrence of $H^{n-1}(\partial B)$ by $P(B)$, the De-Giorgi perimeter.

(2) Show that $\mathcal{H}^{n-1}(\partial A)$ and $Leb^{+}(A)$ are incomparable - either one may be strictly larger than the other.

(3) Calculate the volume of the n-dimensional Euclidean unit-ball.

(4) Show that the isoperimetric inequality on $(\mathbb{R}^n, |\cdot|, Leb)$ implies the Brunn-Minkowski inequality when one of the sets is the Euclidean ball D:

$$\mathrm{Vol}(A+D)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(D)^{1/n} .$$

(5) Let $K = -K$ be a convex, compact set with non-empty interior, and let $\|\cdot\|_K$ be the norm on $\mathbb{R}^n$ whose unit-ball is K. Formulate and prove the (sharp) isoperimetric inequality on $(\mathbb{R}^n, \|\cdot\|_K, Leb)$. Was the assumption that $K = -K$ necessary?
Just for fun, calculate $Leb^{+}_{\|\cdot\|_Q}(D)$, where $Q = [-1,1]^2$ is a cube and D is the Euclidean unit-ball in $\mathbb{R}^2$.

(6) Approximating a general compact set by boxes:
  (a) Let $\mathcal{K}$ denote the collection of all compact subsets of $(\mathbb{R}^n, |\cdot|)$, and let $\mathcal{H}$ denote the Hausdorff distance between two compact sets. Show that Leb is upper semi-continuous on $(\mathcal{K}, \mathcal{H})$: if $A_i \to A$ in $(\mathcal{K}, \mathcal{H})$ then $Leb(A) \geq \limsup_{i\to\infty} Leb(A_i)$.
  (b) We have proved the Brunn-Minkowski inequality for compact sets which may be written as $\cup_{i=1}^{N} B_i$, where B_i are n-dimensional boxes (or rectangles), which are mutually disjoint (up to null-sets). Conclude that the BM inequality is valid for arbitrary compact sets.
  (c) (This was an easy remark in class) Conclude that the BM inequality is valid for arbitrary Borel sets.

(7) Prove that $det^{1/n}$ is concave on the class of $n \times n$ positive semi-definite matrices:

$$A, B \geq 0 \quad \Rightarrow det(A+B)^{1/n} \geq det(A)^{1/n} + det(B)^{1/n} \; .$$

(8) Show that the following three statements are equivalent:

$$\forall A, B \subset \mathbb{R}^n \qquad \mathrm{Vol}(A+B)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}$$

$$\Updownarrow$$

$$\forall A, B \subset \mathbb{R}^n \; \forall \lambda \in [0,1] \qquad \mathrm{Vol}(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda \mathrm{Vol}(A)^{1/n} + (1-\lambda)\mathrm{Vol}(B)^{1/n}$$

$$\Updownarrow$$

$$\forall A, B \subset \mathbb{R}^n \; \forall \lambda \in [0,1] \qquad \mathrm{Vol}(\lambda A + (1-\lambda)B) \geq \mathrm{Vol}(A)^{\lambda}\mathrm{Vol}(B)^{1-\lambda} \; .$$

(you may assume that A, B are always compact to avoid delicate measurability issues).

(9) Use the Brunn–Minkowski inequality to prove the *Prékopa–Leindler inequality*:

Let $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ denote three measurable functions so that for some $\lambda \in (0,1)$:

$$h(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} g(y)^{1-\lambda} \quad \forall x, y \in \mathbb{R}^n \; .$$

Then the following reverse-type Hölder inequality is satisfied:

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{\lambda} \left(\int_{\mathbb{R}^n} g\right)^{1-\lambda} \; .$$

Try to do it using two different methods:
Method 1: Apply Brunn-Minkowski to the level-sets of h, f, g.
Method 2: First prove using the BM inequality that if:

$$h(\lambda x + (1-\lambda)y)^{\frac{1}{k}} \geq \lambda f(x)^{\frac{1}{k}} + (1-\lambda) g(y)^{\frac{1}{k}} \quad \forall x, y \in \mathbb{R}^n$$

for some natural number k, then:

$$\left(\int_{\mathbb{R}^n} h\right)^{\frac{1}{n+k}} \geq \lambda \left(\int_{\mathbb{R}^n} f\right)^{\frac{1}{n+k}} + (1-\lambda) \left(\int_{\mathbb{R}^n} g\right)^{\frac{1}{n+k}} \; .$$

(this is a particular case of the Borell / Brascamp-Lieb inequalities). Conclude the claim by taking $k \to \infty$.

(10) Prove the Brunn–Minkowski inequality in dimension 1 (directly).

(11) Show that if A is compact in $\mathbb{R}^n$ then so is its Steiner symmetrization $S_H A$.

(12) Let K denote a compact convex set with non-empty interior in $\mathbb{R}^n$, let μ_K denote the uniform probability measure on K, and fix a Euclidean structure on $\mathbb{R}^n$. Formulate and prove Brunn's Concavity Principle for k-dimensional marginals of μ_K.

(13) Extend the previous exercise to log-concave functions. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}_+$ is called log-concave if:

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda} \quad \forall x, y \in \mathbb{R}^n \quad \forall \lambda \in (0,1) \ ,$$

or equivalently, if $-\log f : \mathbb{R}^n \to (-\infty, +\infty]$ is convex.

(a) Show that if f is log-concave and integrable, then so are all of its marginals. In other words, show that if $f : \mathbb{R}^{n+m} \to \mathbb{R}_+$ is log-concave and integrable, then so is $h : \mathbb{R}^n \to \mathbb{R}_+$, where $h(x) := \int_{\mathbb{R}^m} f(x,y)dy$.

(b) Deduce from part (a) that if $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are log-concave and integrable, then so is their convolution $f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy$.

(14) **(Not given in class)** Generalize the previous two exercises as follows: Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function so that $f^{1/m}$ is concave on its support (m is a positive integer). Formulate and prove the concavity properties of its k-dimensional marginals.
Hint: apply similar arguments to those required in Method 2 of Exercise 9.

(15) A Borel measure μ on $\mathbb{R}^n$ is called log-concave if:

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for all $\lambda \in (0,1)$ and Borel sets $A, B \subset \mathbb{R}^n$ so that $\lambda A + (1-\lambda)B$ is Borel.

Show that if $f : \mathbb{R}^n \to \mathbb{R}_+$ is log-concave then the measure $\mu := f(x)dx$ is log-concave.

Remark: as mentioned in class, an appropriate converse to this statement was proved by C. Borell.

(16) Let A be a compact set in $\mathbb{R}^n$, and let $S_H A$ denote its Steiner symmetrization about the hyperplane H. Show that A is a Euclidean ball centered at the origin if and only if $S_H A = A$ for all hyperplanes H.

(17) In X. Cabré's proof of the isoperimetric inequality, show that the Euclidean unit-ball is a subset of the set $\nabla v(\Gamma_v)$, where v is the solution to the Neumann problem $\Delta v = |\partial\Omega|/|\Omega|$ in Ω and $\partial v/\partial n = 1$ on $\partial\Omega$, and Γ_v denotes the lower contact set of the graph of v, i.e. :

$$\Gamma_v := \left\{ x \in \Omega \; ; \; v(y) \geq v(x) + \langle \nabla v(x), y - x \rangle \;\; , \;\; \forall v \in \overline{\Omega} \right\} \; .$$

(18) On the canonical sphere (S^n, d, Vol), show the equivalence between the following formulations of the isoperimetric inequality (below A is a compact set and C is a spherical-cap having the same volume as A):

$$Vol(A^d_\varepsilon) \geq Vol(C^d_\varepsilon) \quad \forall \varepsilon > 0 \; ;$$

and:

$$Vol^+_d(A) \geq Vol^+_d(C) \; .$$

(19) Two Point Symmetrization on (S^n, d, Vol): let A be a compact set, let $S_\Phi A$ denote its two-point symmetrization with respect to $\Phi \in S^n$, and let C denote a spherical-cap having the same volume as A. Verify that:
  (a) $(S_\Phi A)_\varepsilon \subset S_\phi(A_\varepsilon)$.
  (b) The set $T = \{B \subset \mathcal{K}(S^n) \; ; \; Vol(B) = Vol(A) \; , \; Vol(B_\varepsilon) \leq Vol(A_\varepsilon)\}$ is closed in $(\mathcal{K}(S^n), \mathcal{H})$ (see exercise 6 to recall definitions).
  (c) The mapping $(\mathcal{K}(S^n), \mathcal{H}) \ni B \mapsto Vol(B \cap C)$ is upper semi-continuous.

(20) Let $f : (S^n, d) \to \mathbb{R}$ be 1-Lipschitz, and let m_f and $\mathbb{E}f$ denote its median and expectation with respect to the uniform measure μ, respectively.
  (a) Show that $|m_f - \mathbb{E}f| \leq \frac{C}{\sqrt{n}}$.
  (b) Show that $\mathbb{E}f \leq \sqrt{\mathbb{E}f^2} \leq \mathbb{E}f + \frac{C}{\sqrt{n}}$.
  (c) Show that $\mu\{|f - \mathbb{E}f| \geq r\} \leq C \exp\left(-\frac{n-1}{2} r^2\right)$ for all $r > 0$.

Here as elsewhere, c, C, etc. are some positive universal numeric constants.

(21) Let $f : (S^n, d) \to \mathbb{R}$ have modulus of continuity:

$$\omega_f(\varepsilon) := \sup\{|f(x) - f(y)| \; ; \; d(x,y) \leq \varepsilon\} \; .$$

Estimate the concentration of f around its median m_f (with respect to the uniform measure μ), i.e. upper bound $\mu\{|f - m_f| \geq r\}$ as a function of $r > 0$. In particular, what is the concentration of an L-Lipschitz function?

(22) Let A denote a $k \times n$ random matrix with i.i.d. standard Gaussian entries. Show that with very high-probability (quantify this!), the map $T : \mathbb{R}^n \to \mathbb{R}^k$ given by $T(x) := \frac{1}{\sqrt{k}} Ax$ is a good "Johnson-Lindenstrauss" map, i.e.:

$$(1-\varepsilon)|z|_{\mathbb{R}^n} \leq |T(z)|_{\mathbb{R}^k} \leq (1+\varepsilon)|z|_{\mathbb{R}^n}$$

with very high-probability for a fixed $z \in \mathbb{R}^n$.

(23) Prove Bobkov's 3-point inequality:

$$I\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(\sqrt{I(a)^2 + \left(\frac{b-a}{2}\right)^2} + \sqrt{I(b)^2 + \left(\frac{b-a}{2}\right)^2}\right) \quad , \; \forall a,b \in [0,1] \; .$$

Here I denotes the Gaussian isoperimetric profile, i.e. $I = \phi \circ \Phi^{-1}$, where ϕ denotes the standard one-dimensional Gaussian density, and Φ is its cumulative distribution function.

(24) Formulate and prove the sharp isoperimetric inequality for the space $(\mathbb{R}^k, \|\cdot\|, \gamma_k)$, where γ_k is the k-dimensional standard Gaussian measure and $\|\cdot\|$ is a norm on $\mathbb{R}^k$ having unit ball $K = \{x; \|x\| \leq 1\}$.

(25) Let T denote a ℓ-dimensional Ehrhard symmetrization parallel to E in the direction of $u \in E$ (as defined in class).
  (a) Show that if A is closed then so is $T(A)$.
  (b) Obviously $\gamma_k(T(A)) = \gamma_k(A)$ by Fubini's theorem.
  (c) Show that $(T(A))_\varepsilon \subset T(A_\varepsilon)$ for any extension $\varepsilon > 0$, using the Gaussian isoperimetric inequality in dimension ℓ.
  (d) Deduce that $\gamma_k^+(T(A)) \leq \gamma_k^+(A)$.

(26) Prove Ehrhard's inequality for convex sets:
  (a) Optional: prove Ehrhard's inequality for convex sets in dimension 1.

(b) Use Ehrhard's inequality in dimension ℓ to show that if A is convex then so is $T(A)$, for any ℓ-dimensional Ehrhard symmetrization T (we will only need this for $\ell = 1$ below).
(c) Use 1-symmetrizations in $\mathbb{R}^{k+1}$ to prove Ehrhard's inequality for convex sets in $\mathbb{R}^k$. Hint: recall the proof of Brunn's concavity principle.

(27) Prove that among all Borel sets in $\mathbb{R}^3$ having boundary $S^1 \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, the Euclidean ball $B^2 \subset \mathbb{R}^2 \times \{0\}$ has minimal 2-dimensional volume.

(28) Prove that in the Euclidean case, if N is a smooth $n-1$ dimensional hypersurface oriented by a unit normal vector field ν, then the following two definitions for the second fundamental form $II^{\nu}_{N,x}$ at $x \in N$ coincide:
(a) $II^{\nu}_{N,x}$ is the symmetric bi-linear form on T_xN given by $Hessf$, when locally around x we may write $N = \{y - f(y)\nu \; ; \; y \in T_xN\}$ for a function $f : T_xN \to \mathbb{R}$, with T_xN being identified with the affine hyperplane $x + \nu^{\perp}$.
(b) $II^{\nu}_{N,x} = \nabla \nu|_{T_xN}$.

(29) The final step of the Lévy-Gromov Theorem: assume that for some $v \in (0,1)$ and $k > 0$:

$$\begin{aligned} 1 - v &\leq A \int_0^{\infty} J_{k,H}(t)dt \;, \\ v &\leq A \int_0^{\infty} J_{k,-H}(t)dt \;, \end{aligned}$$

where:

$$J_{k,H}(t) = \left(s'_k(t) + \frac{H}{n-1} s_k(t) \right)_+^{n-1} \;, \; s_k(t) = \frac{\sin(\sqrt{k}t)}{\sqrt{k}}$$

(here $n > 1$ is some fixed integer). Conclude that:

$$A \geq \varphi \circ \Phi^{-1}(v) \;,$$

where:

$$\varphi(t) = \frac{\sin^{n-1}(\sqrt{k}t)}{\int_0^{\pi/\sqrt{k}} \sin^{n-1}(\sqrt{k}s)ds} \;, \; \Phi(t) = \int_0^t \varphi(s)ds \;.$$

(30) Recall that the tangent space to $SO(n)$ at Id may be identified with anti-symmetric n by n matrices. We equip it with the Riemannian metric induced from Euclidean space $\mathbb{R}^{n^2}$.
(a) Show that the dimension of $SO(n)$ is $n(n-1)/2$.
(b) Calculate a sharp lower bound on the Ricci curvature of $SO(n)$. Use that the sectional-curvature of the 2-plane spanned by two orthogonal anti-symmetric matrices B_1 and B_2 is equal to $1/4 \left\|[B_1, B_2]\right\|^2$, where $[B_1, B_2] = B_1B_2 - B_2B_1$ and $\|\cdot\|$ denotes the Hilbert-Schmidt norm (the norm induced from $\mathbb{R}^{n^2}$).

(31) Given $v \in [0,1]$, find a minimizer of $|\partial A|$ among all Borel subsets of $[0,1]^2$ of volume v. Use the fact that $\partial A \cap (0,1)^2$ is smooth and has constant mean-curvature, and that ∂A meets the boundary of $[0,1]^2$ tangentially.

(32) Given a mm-space (Ω, d, μ), prove a log-Sobolev inequality implies a Spectral-Gap inequality. Specifically, prove that $\lambda_{SG} \geq \rho_{LS}$, by using $f = 1+\varepsilon h$ with $\int h d\mu = 0$ in the log-Sobolev inequality, and taking the limit as $\varepsilon \to 0$.

(33) Formulate and prove the tensorization property of the Spectral-Gap inequality (hint: recall the tensorization property of Bobkov's inequality seen in class).

(34) Given (Ω, d, μ) with $\mu(\Omega) = 1$, prove that for all $p \geq 1$:

$$\frac{1}{2} \|f - E_\mu f\|_{L^p(\mu)} \leq \|f - med_\mu f\|_{L^p(\mu)} \leq 3 \|f - E_\mu f\|_{L^p(\mu)} \ ,$$

where $E_\mu f = \int f d\mu$ and $med_\mu f$ denotes a median of f with respect to μ, i.e. a median of the push-forward of μ via f.
Hint: start with $| \|f - E_\mu f\| - \|f - med_\mu f\| | \leq |E_\mu f - med_\mu f| \leq \ldots$.

(35) Prove the Sobolev-Gagliardo inequalities in Euclidean space $(\mathbb{R}^n, |\cdot|)$ (with non-sharp constant):

$$\||\nabla g|\|_{L^q} \geq \frac{n}{n-1}\frac{c_n}{p} \|g\|_{L^p} \quad \forall \text{ compactly supported Lipschitz } g \ ,$$

where $1 \leq q < n$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, and c_n denotes the sharp constant in the Euclidean isoperimetric inequality: $|\partial A| \geq c_n |A|^{\frac{n-1}{n}}$.
Hint: translate the isoperimetric inequality into its functional form $q = 1$ above. Then substitute $g = f^r$, and conclude by applying Hölder's inequality.

(36) Given a mm-space (Ω, d, μ), prove that a Gaussian isoperimetric inequality $\mathcal{I} \geq D\mathcal{I}_\gamma$ (here $\mathcal{I}_\gamma$ denotes the Gaussian isoperimetric profile and $D > 0$) implies a log-Sobolev inequality:

$$\frac{D^2}{2}\mathrm{Ent}_\mu(f^2) \leq \int |\nabla f|^2 d\mu \quad \forall \text{ Lipschitz } f \ .$$

Hint: assume that f is bounded. Pass to Bobkov's functional version of the Gaussian isoperimetric inequality:

$$\int \sqrt{|\nabla g|^2 + (D\mathcal{I}_\gamma(g))^2} d\mu \geq D\mathcal{I}_\gamma\Big(\int g d\mu\Big) \quad \forall \text{ Lipschitz } g : \Omega \to [0,1] \ ,$$

apply to $g = \varepsilon f^2$, take the limit as $\varepsilon \to 0$ and use that $\mathcal{I}_\gamma(v) = \sqrt{2}v\sqrt{\log 1/v} + o(v)$ as $v \to 0$.

Additional Reading

- "Geometric inequalities" - Burago and Zalgaller.
- "The Concentration of Measure Phenomenon" - M. Ledoux.
- "Geometric Measure Theory - A beginner's guide" - F. Morgan.
- "Riemmanian Geometry" - Gallot, Hulin and Lafontaine.
- "Riemannian Geometry" - do Carmo.
- "Isoperimetric Inequalities: Differential geometric and analytic perspectives" - I. Chavel.
- "Eigenvalues in Riemannian Geometry" - I. Chavel.