

**ISOPERIMETRY, SOBOLEV AND CONCENTRATION
INEQUALITIES THROUGH THE LENS OF CONVEXITY**

(note that the statement of one exercise may be used for the proof of another).

- (1) Prove that $\det^{1/n}$ is concave on the class of $n \times n$ positive semi-definite matrices:

$$A, B \geq 0 \quad \Rightarrow \quad \det(A + B)^{1/n} \geq \det(A)^{1/n} + \det(B)^{1/n} .$$

- (2) Recall that the collection \mathcal{K} of all compact subsets of a complete separable metric space (Ω, d) , equipped with the Hausdorff distance $\mathcal{H}(K_1, K_2) = \inf \{ \varepsilon > 0 ; K_1 \subset (K_2)_\varepsilon \text{ and } K_2 \subset (K_1)_\varepsilon \}$, is a complete metric space. Also, recall that a Borel measure μ on (Ω, d) is called inner regular if for any Borel set A :

$$\mu(A) = \sup \{ \mu(K) ; A \supset K \text{ compact} \} ,$$

and outer regular if:

$$\mu(A) = \inf \{ \mu(G) ; A \subset G \text{ open} \} .$$

Recall that the Lebesgue measure Vol on Euclidean space is both inner and outer regular.

- (a) Show that any outer regular measure μ is upper semi-continuous on $(\mathcal{K}, \mathcal{H})$:

$$K_i \rightarrow_{\mathcal{H}} K \quad \Rightarrow \quad \mu(K) \geq \limsup_{i \rightarrow \infty} \mu(K_i) .$$

- (b) We have shown in class the Brunn–Minkowski (BM) inequality for sets A, B which are unions of axis-aligned boxes. Deduce the BM inequality for all compact sets $A, B \subset \mathbb{R}^n$.
- (c) Deduce the BM inequality for all Borell sets $A, B \subset \mathbb{R}^n$.
- (3) Recall that the Prékopa–Leindler (PL) inequality states that if $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ denote three measurable functions so that for some $\lambda \in (0, 1)$:

$$h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda} \quad \forall x, y \in \mathbb{R}^n .$$

Then:

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f \right)^\lambda \left(\int_{\mathbb{R}^n} g \right)^{1-\lambda} .$$

We've seen in class that the n -D PL inequality implies the n -D BM inequality. The goal of this exercise is to show the reverse implication. We've already seen that in class that the 1-D BM inequality implies the 1-D PL inequality. Generalize this to arbitrary dimension n using two different methods:

(a) Method 1. Apply the BM inequality in \mathbb{R}^n to the level-sets of h, f, g , and conclude using the 1-D PL inequality.

(b) Method 2. First prove, using the BM inequality in \mathbb{R}^{n+k} , that if:

$$h(\lambda x + (1 - \lambda)y)^{\frac{1}{k}} \geq \lambda f(x)^{\frac{1}{k}} + (1 - \lambda)g(y)^{\frac{1}{k}} \quad \forall x, y \in \mathbb{R}^n$$

for some natural number k , then:

$$\left(\int_{\mathbb{R}^n} h \right)^{\frac{1}{n+k}} \geq \lambda \left(\int_{\mathbb{R}^n} f \right)^{\frac{1}{n+k}} + (1 - \lambda) \left(\int_{\mathbb{R}^n} g \right)^{\frac{1}{n+k}}.$$

(do not use the statement of other exercises for the proof). Conclude the n -D PL inequality by taking $k \rightarrow \infty$ and using that for $a, b > 0$:

$$\lim_{\varepsilon \rightarrow 0} (\lambda a^\varepsilon + (1 - \lambda)b^\varepsilon)^{1/\varepsilon} \rightarrow a^\lambda b^{1-\lambda}.$$

(4) C. Borell generalized the BM/PL inequalities as follows: let $\Psi_\lambda : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ denote a continuous, increasing in each variable, and 1-homogeneous function (i.e. $\Psi_\lambda(tx, ty) = t\Psi_\lambda(x, y) \quad \forall t > 0$). Let $f_0, f_1, f_\lambda \geq 0$ denote continuous integrable functions on \mathbb{R}^n , so that $\int f_0, \int f_1 > 0$. Then the following statements are equivalent:

(a) For all compact $A_0, A_1 \subset \mathbb{R}^n$:

$$\int_{(1-\lambda)A_0 + \lambda A_1} f_\lambda \geq \Psi_\lambda \left(\int_{A_0} f_0, \int_{A_1} f_1 \right).$$

(b) For all $x_0, x_1 \in \mathbb{R}^n$, for all $a^0, a^1 \in \mathbb{R}_+^n$ (the positive orthant):

$$f_\lambda((1 - \lambda)x_0 + \lambda x_1) \prod_{i=1}^n ((1 - \lambda)a_i^0 + \lambda a_i^1) \geq \Psi_\lambda(f_0(x_0) \prod_{i=1}^n a_i^0, f_1(x_1) \prod_{i=1}^n a_i^1).$$

Note that the function $\Psi_\lambda(x, y) = ((1 - \lambda)x^{1/n} + \lambda y^{1/n})^n$ corresponds to the BM inequality, whereas the function $\Psi_\lambda(x, y) = x^{1-\lambda}y^\lambda$ corresponds to the PL inequality.

The proof that (a) implies (b) is immediate by testing (a) on small axis-aligned boxes. Assuming that (b) implies (a) indeed holds in dimension 1, show that (b) implies (a) holds in any dimension n , using two different methods:

(a) Induction on the dimension n .

(b) Proving (a) for sets A_0, A_1 which are unions of k_0, k_1 small axis-aligned boxes (respectively) having disjoint interiors, using induction on $k_0 + k_1$.

(5) Using Borell's theorem in the previous exercise, find the optimal function $\Phi_{N,n,\lambda} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, so that for all f_0, f_1, f_λ as above satisfying:

$$f_\lambda((1 - \lambda)x_0 + \lambda x_1) \geq \Phi_{N,n,\lambda}(f_0(x_0), f_1(x_1)) \quad \forall x_0, x_1 \in \mathbb{R}^n,$$

it follows that:

$$\int f_\lambda \geq \Phi_{N,n,\lambda} \left(\int f_0, \int f_1 \right),$$

where $\Psi_{N,\lambda} = ((1-\lambda)x^{1/N} + \lambda y^{1/N})^N$ for some fixed $N \geq n$. The resulting inequality is called the Borell / Brascamp–Lieb inequality (compare with exercise 3 (b)).

- (6) We've seen in class that if $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a log-concave function:

$$\forall \lambda \in [0, 1] \quad \forall x, y \in \mathbb{R}^n \quad f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda},$$

then $\mu = f(x)dx$ is a log-concave measure:

$$\forall A, B \subset \mathbb{R}^n \quad \mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Show the converse.

- (7) Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$. Show that if $0 < p < q$, then the property “ $f^q(t)$ is concave on its support” implies “ $f^p(t)$ is concave on its support”. Passing to the limit $p \rightarrow 0$, show that this implies that $\log f(t) : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ is concave (i.e. that f is log-concave).
- (8) Show that if A is compact in \mathbb{R}^n then so is its Steiner symmetrization $S_H A$.
- (9) Let A be a compact subset of \mathbb{R}^n . Show that $S_H A = A$ for all centered hyperplanes H , if and only if A is a centered Euclidean ball.
- (10) Let $K \subset \mathbb{R}^n$ be a convex body, and let E be an m -dimensional subspace. Show that the function:

$$E \ni y \mapsto \text{Vol}(K \cap (y + E^\perp))^{\frac{1}{n-m}},$$

is concave on its support.

- (11) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function so that $f^{1/k}$ is concave on its support, for some integer k , and let E denote an m -dimensional subspace. Denoting:

$$g(y) := \int_{y+E^\perp} f(x)dx,$$

show that the function:

$$E \ni y \mapsto g(y)^{\frac{1}{n+k-m}},$$

is concave on its support (Hint: construct an appropriate convex set in \mathbb{R}^{n+k}). Taking $k \rightarrow \infty$, deduce (what we have already seen in class) that when f is log-concave, then so is its marginal g .

- (12) Show that the isoperimetric inequality on \mathbb{R}^n :

$$|\partial A| \geq n |B_2^n|^{1/n} |A|^{(n-1)/n},$$

implies the Brunn-Minkowski inequality when one of the sets is a Euclidean ball D :

$$\text{Vol}(A + D)^{1/n} \geq \text{Vol}(A)^{1/n} + \text{Vol}(D)^{1/n},$$

(say for sets A with smooth boundary).

- (13) Let $K = -K$ be an origin-symmetric convex body in \mathbb{R}^n and E a linear subspace of dimension k . Using exercise 10, prove the Rogers–Shephard inequality:

$$|K| \leq \left| K \cap E^\perp \right| |Proj_E K| \leq \binom{n}{k} |K| .$$

- (14) Prove that $\mathbb{R}_+ \ni r \mapsto Vol(K \cap rD_n)^{1/n}$ is a concave function, for any convex K in \mathbb{R}^n .
 (15) Recall that Steiner’s formula states that for all $t \geq 0$:

$$Vol(K + tB_2^n) = \sum_{i=0}^n \binom{n}{i} W_{n-i}(K) t^i .$$

Let K be a smooth convex body. Prove using Steiner’s formula and integration in polar coordinates that $W_1(K) = Vol(B_2^n)W(K)$, where recall:

$$W(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) .$$

Here σ is the uniform probability measure on S^{n-1} and $h_K(\theta) = \sup \{ \langle x, \theta \rangle ; x \in K \}$ is the support function of K .

- (16) Recall Minkowski’s generalization of Steiner’s theorem, stating that for any convex sets $K_1, \dots, K_m \subset \mathbb{R}^n$ and $t_1, \dots, t_m \geq 0$:

$$Vol\left(\sum_{i=1}^m t_i K_i\right) = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq m} V(K_{i_1}, \dots, K_{i_n}) t_{i_1} \cdots t_{i_n},$$

where $V(L_1, \dots, L_n)$ is the mixed volume of the convex sets $L_1, \dots, L_n \subset \mathbb{R}^n$ - it is chosen to be invariant under permutations, and is multi-linear with respect to Minkowski addition in each variable. It is a non-trivial fact that it is always non-negative.

We use the notation $V(K; i, L; n - i)$ to denote the mixed volume of the tuple where K appears i times and L appears $n - i$ times. Comparing with Steiner’s theorem, we see that $W_i(K) = V(K; i, B_2^n; n - i)$.

The deep Alexandrov–Fenchel inequality states that:

$$V(K_1, K_2, L_3, \dots, L_n)^2 \geq V(K_1, K_1, L_3, \dots, L_n) V(K_2, K_2, L_3, \dots, L_n) .$$

Using only this, deduce:

- (a) $V(K; i, L; n - i) \geq Vol(K)^{i/n} Vol(L)^{(n-i)/n}$.
 (b) More generally, deduce Minkowski’s inequality:

$$V(K_1, \dots, K_n) \geq (\prod_{i=1}^n Vol(K_i))^{1/n} .$$

- (c) Specializing (a) to the case $i = n - 1$, we have Minkowski’s inequality:

$$V(K; n - 1, L; 1) \geq Vol(K)^{(n-1)/n} Vol(L)^{1/n} .$$

Note and explain why the case of $L = B_2^n$ corresponds to the Euclidean isoperimetric inequality (for convex bodies).

- (d) Use the inequality in (c) above and the multi-linearity of the mixed-volumes V to deduce the Brunn–Minkowski inequality:

$$\text{Vol}(K + L)^{1/n} \geq \text{Vol}(K)^{1/n} + \text{Vol}(L)^{1/n} .$$

- (e) Generalize all of the above to show that for all $i = 1, \dots, n$:

$$W_i(K + L)^{1/i} \geq W_i(K)^{1/i} + W_i(L)^{1/i} , \quad \forall i = 1, \dots, n .$$

- (f) Show that for all $i = 1, \dots, n - 1$:

$$W_i(K)^2 \geq W_{i-1}(K)W_{i+1}(K),$$

and deduce that $i \mapsto (W_i(K)/\text{Vol}(B_2^n))^{1/i}$ is a non-increasing sequence (note that $W_0(K) = \text{Vol}(B_2^n)$).

- (17) Calculate the volume of B_p^n , the unit-ball of ℓ_p^n , by integrating the measure $\exp(-\|x\|_{\ell_p^n}^p)dx$ on \mathbb{R}^n .

- (18) Consider the metric-measure space (S^n, d, μ_{S^n}) , where $\mu = \mu_{S^n}$ is the corresponding Haar probability measure, and d is the geodesic distance on S^n . Let $f : S^n \rightarrow \mathbb{R}$ be a 1-Lipschitz function, let $m(f)$ denote its median, and define $E(f) = \int f d\mu$.

- (a) Show that $|E(f) - m(f)| \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
 (b) Show that $0 \leq \sqrt{E(f^2)} - E(|f|) \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
 (c) Deduce concentration around $E(f)$, and if $f \geq 0$, also around $\sqrt{E(f^2)}$. In other words, show that:

$$\mu(x \in S^n ; |f(x) - A_f| \geq r) \leq C \exp\left(-\frac{n-1}{2}r^2\right) ,$$

where A_f is either $E(f)$, and when $f \geq 0$, also $\sqrt{E(f^2)}$, for some constant $C > 0$.

- (d) Show that if f is L -Lipschitz, then:

$$\mu(x \in S^n ; f(x) - m(f) \geq r) \leq \sqrt{\frac{\pi}{8}} \exp\left(-\frac{n-1}{2L^2}r^2\right) .$$

- (e) More generally, show that:

$$\mu(x \in S^n ; f(x) - m(f) \geq \omega_f(r)) \leq \sqrt{\frac{\pi}{8}} \exp\left(-\frac{n-1}{2}r^2\right) ,$$

where $\omega_f(r) = \sup\{|f(x) - f(y)| ; d(x, y) \leq r\}$ denotes the modulus of continuity of f .

- (19) Show that $\mu_{S^n}(B_\theta(x_0)) \geq c\sqrt{n-1} \theta \sin^{n-1}(\theta)$, where $B_\theta(x_0)$ is a geodesic ball of radius θ on S^n , $x_0 \in S^n$ is any point, and $c > 0$ is a constant.

- (20) Show that the Gaussian isoperimetric inequality on $(\mathbb{R}^n, |\cdot|, \gamma_n)$, where γ_n denotes the standard Gaussian measure on \mathbb{R}^n , namely:

$$\gamma_n(A) = \gamma_n(H) \Rightarrow \gamma_n^+(A) \geq \gamma_n^+(H),$$

implies:

$$\gamma_n(A) = \gamma_n(H) \Rightarrow \gamma_n(A_r) \geq \gamma_n(H_r), \forall r > 0.$$

Here H is (any) half-plane, $\gamma_n^+(C)$ denote the Gaussian boundary measure of a Borel set $C \subset \mathbb{R}^n$, and C_r denote the r -extension of C .

- (21) Let A denote a $k \times n$ random matrix with i.i.d. standard Gaussian entries. Show that with very high-probability (quantify this!), the map $T : \mathbb{R}^n \rightarrow \mathbb{R}^k$ given by $T(x) := \frac{1}{\sqrt{k}}Ax$ is a good “Johnson-Lindenstrauss” map, i.e.:

$$(1 - \varepsilon)|z|_{\mathbb{R}^n} \leq |T(z)|_{\mathbb{R}^k} \leq (1 + \varepsilon)|z|_{\mathbb{R}^n}$$

with very high-probability for a fixed $z \in \mathbb{R}^n$.

- (22) Assume that on a metric-measure (probability) space (Ω, d, μ) we know that:

$$\forall A \subset \Omega \quad \mu(A) \geq 1/2 \Rightarrow \mu(\Omega \setminus A_r) < K(r) \quad \forall r > 0.$$

Show that:

$$\forall A \subset \Omega \quad \mu(A) \geq K(\varepsilon_0) \Rightarrow \mu(\Omega \setminus A_{r+\varepsilon_0}) < K(r) \quad \forall r > 0.$$

Deduce that on S^n , we have the following concentration property:

$$\mu(A) \geq \sqrt{\pi/8} \exp\left(-\frac{n-1}{2}\varepsilon_0^2\right) \Rightarrow \mu(S^n \setminus A_{2\varepsilon_0}) \leq \sqrt{\pi/8} \exp\left(-\frac{n-1}{2}\varepsilon_0^2\right).$$

- (23) Two Point Symmetrization on (S^n, d, Vol) : let A be a compact set, let $S_\Phi A$ denote its two-point symmetrization with respect to $\Phi \in S^n$, and let C denote a spherical-cap having the same volume as A . Verify that:

- $Vol(S_\Phi(A)) = Vol(A)$
- $S_\Phi(A)$ is closed.
- $(S_\Phi A)_\varepsilon \subset S_\Phi(A_\varepsilon)$.
- $S_\Phi(C)$ is another spherical-cap.
- The set $T = \{B \subset \mathcal{K}(S^n) ; Vol(B) = Vol(A), Vol(B_\varepsilon) \leq Vol(A_\varepsilon)\}$ is closed in $(\mathcal{K}(S^n), \mathcal{H})$ (see exercise 2 to recall definitions).
- The mapping $(\mathcal{K}(S^n), \mathcal{H}) \ni B \mapsto Vol(B \cap C)$ is upper semi-continuous.

- (24) Prove Bobkov’s 3-point inequality:

$$I\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left(\sqrt{I(a)^2 + \left(\frac{b-a}{2}\right)^2} + \sqrt{I(b)^2 + \left(\frac{b-a}{2}\right)^2} \right), \quad \forall a, b \in [0, 1].$$

Here I denotes the Gaussian isoperimetric profile, i.e. $I = \phi \circ \Phi^{-1}$, where ϕ denotes the standard one-dimensional Gaussian density, and Φ is its cumulative distribution function.

- (25) Formulate and prove the sharp isoperimetric inequality for the space $(\mathbb{R}^k, \|\cdot\|, \gamma_k)$, where γ_k is the k -dimensional standard Gaussian measure and $\|\cdot\|$ is a norm on \mathbb{R}^k having unit ball $K = \{x; \|x\| \leq 1\}$.

Guidance: repeat the argument of deducing the usual Gaussian isoperimetric inequality starting from Ehrhard's inequality.

- (26) Given a mm-space (Ω, d, μ) , prove a log-Sobolev inequality implies a Spectral-Gap inequality. Specifically, prove that $\lambda_{SG} \geq \rho_{LS}$, by using $f = 1 + \varepsilon h$ with $\int h d\mu = 0$ in the log-Sobolev inequality, and taking the limit as $\varepsilon \rightarrow 0$.

- (27) Formulate and prove the tensorization property of the Spectral-Gap inequality (hint: recall the tensorization property of Bobkov's inequality seen in class).

- (28) Given (Ω, d, μ) with $\mu(\Omega) = 1$, prove that for all $p \geq 1$:

$$\frac{1}{2} \|f - E_\mu f\|_{L^p(\mu)} \leq \|f - \text{med}_\mu f\|_{L^p(\mu)} \leq 3 \|f - E_\mu f\|_{L^p(\mu)} ,$$

where $E_\mu f = \int f d\mu$ and $\text{med}_\mu f$ denotes a median of f with respect to μ , i.e. a median of the push-forward of μ via f .

Hint: start with $|\|f - E_\mu f\| - \|f - \text{med}_\mu f\|| \leq |E_\mu f - \text{med}_\mu f| \leq \dots$

- (29) Prove the Sobolev-Gagliardo inequalities in Euclidean space $(\mathbb{R}^n, |\cdot|)$ (with non-sharp constant):

$$\|\nabla g\|_{L^q} \geq \frac{n}{n-1} \frac{c_n}{p} \|g\|_{L^p} \quad \forall \text{ compactly supported Lipschitz } g ,$$

where $1 \leq q < n$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, and c_n denotes the sharp constant in the Euclidean isoperimetric inequality: $|\partial A| \geq c_n |A|^{\frac{n-1}{n}}$.

Hint: translate the isoperimetric inequality into its functional form $q = 1$ above. Then substitute $g = f^r$, and conclude by applying Hölder's inequality.

- (30) Given a mm-space (Ω, d, μ) , prove that a Gaussian isoperimetric inequality $\mathcal{I} \geq D\mathcal{I}_\gamma$ (here \mathcal{I}_γ denotes the Gaussian isoperimetric profile and $D > 0$) implies a log-Sobolev inequality:

$$\frac{D^2}{2} \text{Ent}_\mu(f^2) \leq \int |\nabla f|^2 d\mu \quad \forall \text{ Lipschitz } f .$$

Hint: assume that f is bounded. Pass to Bobkov's functional version of the Gaussian isoperimetric inequality:

$$\int \sqrt{|\nabla g|^2 + (D\mathcal{I}_\gamma(g))^2} d\mu \geq D\mathcal{I}_\gamma\left(\int g d\mu\right) \quad \forall \text{ Lipschitz } g : \Omega \rightarrow [0, 1] ,$$

apply to $g = \varepsilon f^2$, take the limit as $\varepsilon \rightarrow 0$ and use that $\mathcal{I}_\gamma(v) = \sqrt{2}v\sqrt{\log 1/v} + o(v)$ as $v \rightarrow 0$.

- (31) Use Herbst's method to show that if the following log-Sobolev inequality holds for some $\rho > 0$:

$$\rho \text{Ent}_\mu(g) \leq \int \frac{|\nabla g|^2}{g} d\mu,$$

for all nice $g > 0$, and if f is 1-log-Lipschitz: $|\nabla \log f| \leq 1$, then f has comparable $L^p(\mu)$ moments:

$$\|f\|_{L^q(\mu)} \leq \exp\left(\frac{q-p}{\rho}\right) \|f\|_{L^p(\mu)} \quad \forall 0 < p < q.$$

- (32) Recall that the spherically symmetric decreasing rearrangement of a measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is defined as the function $f^* : \mathbb{R}^n \rightarrow \mathbb{R}_+$ given by:

$$f^*(x) = \sup \{s \geq 0; \text{Vol} \{f \geq s\} \geq \text{Vol}(B(0, |x|))\},$$

where $B(0, a)$ denotes the ball centered at the origin of radius a . Show that:

$$\text{Vol} \{f^* \geq t\} = \text{Vol} \{f \geq t\} \quad \forall t \geq 0.$$

- (33) Prove the spherical Faber-Krahn inequality: for any $\Omega \subset S^n$, one has the following inequality for the first eigenvalue of the Laplacian with zero Dirichlet boundary conditions on the corresponding domains:

$$\lambda_1^D(\Omega) \geq \lambda_1^D(\Omega^*),$$

where Ω^* denotes the geodesic ball having the same volume as Ω .

- (34) Let $\Delta_\mu = \Delta - \langle \nabla V, \nabla \rangle$ denote the weighted Laplacian operator on the Hilbert space $H = L^2(\mu)$, with invariant measure $\mu = \exp(-V(x))dx$ on \mathbb{R}^n . Recall That $-\Delta_\mu$ is symmetric and positive semi-definite on the domain $D := C_c^\infty(\mathbb{R}^n)$, the set of compactly supported smooth functions on \mathbb{R}^n . The purpose of this exercise is to outline how to prove that $-\Delta_\mu$ is essentially self-adjoint (E.S.A.) on the latter domain D . As known, being positive semi-definite, this is equivalent to showing that $\overline{(-\Delta_\mu + Id)}(D) = H$. Assume that there exists $u \in H$ so that:

$$\langle -\Delta_\mu f, u \rangle + \langle f, u \rangle = 0 \quad \forall f \in D.$$

In other words, $\Delta_\mu u = u$ weakly. Sobolev embedding theorems and elliptic regularity consequently imply that in fact $u \in C_{loc}^\infty(\mathbb{R}^n)$, so that $\Delta_\mu u = u$ in the classical sense. Show that this necessarily implies $u = 0$, yielding the E.S.A. of $-\Delta_\mu$.

Guidance: construct a family of functions $f_k \geq 0$ in D pointwise converging to the constant function 1 on \mathbb{R}^n with $\|\nabla f_k\|_{L^\infty} \leq 1/k$. Evaluate:

$$\int f_k^2 u \Delta_\mu u d\mu,$$

by integrating-by-parts, and conclude that:

$$\int f_k^2 |\nabla u|^2 d\mu \leq \frac{4}{k^2} \int u^2 d\mu.$$

Letting $k \rightarrow \infty$, conclude that $u = 0$, as asserted.

Remark: the same proof applies to any complete weighted manifold.

- (35) Calculate the spectrum of the Laplacian for both the Neumann and Dirichlet boundary conditions on the n -dimensional box $[0, a_1] \times \dots \times [0, a_n]$.
- (36) Let P_t denote a reversible Markov diffusion semi-group on $L^2(\mu)$ where μ is a probability measure. Recall that μ is called invariant for P_t if $\int P_t f d\mu = \int f d\mu$ for all $f \in L^2(\mu)$ and $t \geq 0$. Recall that μ is called ergodic if $P_t f \rightarrow_{t \rightarrow \infty} \int f d\mu$ in $L^2(\mu)$ for any $f \in L^2(\mu)$.
- Show that an invariant measure does not have to be unique in general.
 - Show that if μ is ergodic then μ is the unique invariant measure for P_t .
 - Give an example of a Markov process (or alternatively, of P_t), having a unique invariant measure μ which is not ergodic.

- (37) Verify the log-Sobolev inequalities on the two-point space $\{-1, 1\}$:

$$\forall a, b \in \mathbb{R} \quad \frac{1}{2} (a^2 \log(a^2) + b^2 \log b^2) - \frac{a^2 + b^2}{2} \log \frac{a^2 + b^2}{2} \leq \frac{1}{2} (b - a)^2.$$

- (38) Let $f(x) = f(x_1, \dots, x_n)$ denote a measurable function on the product measure space $(\Omega_1 \times \dots \times \Omega_n, \Sigma_1 \times \dots \times \Sigma_n, \mu := \mu_1 \otimes \dots \otimes \mu_n)$. Given $x = (x_1, \dots, x_n)$, denote $f_i^x(y_i) := f(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ a function on $(\Omega_i, \Sigma_i, \mu_i)$. Prove the sub-additivity of entropy:

$$Ent_\mu(f) \leq \sum_{i=1}^n \int Ent_{\mu_i}(f_i^x) d\mu(x).$$

Guidance: recall that in general:

$$(1) \quad Ent_\nu(h) = \sup \left\{ \int h g d\nu ; \int \exp(g) d\nu \leq 1 \right\}.$$

Given g on the product space with $\int \exp(g) d\mu = 1$, decompose $g = \sum_{i=1}^n g^i$ by defining for $x = (x_1, \dots, x_n)$:

$$\exp(g^i(x)) = \frac{\int \exp(g(x)) d\mu_1(x_1) \dots d\mu_{i-1}(x_{i-1})}{\int \exp(g(x)) d\mu_1(x_1) \dots d\mu_{i-1}(x_{i-1}) d\mu_i(x_i)}$$

(so that the function $g^i(x)$ actually only depends on (x_i, \dots, x_n)). Verify that indeed $g = \sum_{i=1}^n g^i$, that $\int \exp((g^i)^x(y_i)) d\mu_i(y_i) \equiv 1$ for all x , and use this together with formula (1) above to conclude the claim.

(39) Given a reversible semi-group P_t with respect to an invariant measure μ on \mathbb{R}^n , the heat-kernel $k_t : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ is defined to be the function $k_t(x, y)$ (if it exists) so that $P_t f(x) = \int f(y)k_t(x, y)d\mu(y)$ for all f and $x \in \mathbb{R}^n$.

(a) Let γ denote the standard Gaussian probability measure on \mathbb{R}^n . Verify that the Mehler formula for the Ornstein-Uhlenbeck (O-U) semi-group:

$$P_t f(x) = \int f(\exp(-t)x + \sqrt{1 - \exp(-2t)}y)d\gamma(y),$$

indeed solves the diffusion equation:

$$\frac{d}{dt}P_t f = LP_t f,$$

for the O-U generator $Lf = \Delta f(x) - \langle x, \nabla f(x) \rangle$.

(b) Derive a formula for the O-U heat-kernel $k_t(x, y)$.

(c) For the standard heat equation on \mathbb{R}^n , $\frac{d}{dt}P_t f = \Delta P_t f$ with respect to the invariant Lebesgue measure dx , verify that the heat-kernel is given by:

$$k_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{(x - y)^2}{4t}\right).$$

(40) Recall that the tangent space to $SO(n)$ at Id may be identified with anti-symmetric n by n matrices. We equip it with the Riemannian metric induced from Euclidean space \mathbb{R}^{n^2} .

(a) Show that the dimension of $SO(n)$ is $n(n - 1)/2$.

(b) Calculate a sharp lower bound on the Ricci curvature of $SO(n)$. Use that the sectional-curvature of the 2-plane spanned by two orthogonal anti-symmetric matrices B_1 and B_2 is equal to $\frac{1}{4} \|[B_1, B_2]\|_{HS}^2$, where $[B_1, B_2] = B_1 B_2 - B_2 B_1$ and $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm (the norm induced from \mathbb{R}^{n^2}).

(41) Recall that Lichnerowicz' spectral-gap estimate on the canonical unit-sphere $S^n \subset \mathbb{R}^{n+1}$ gives $\lambda_{SG}(S^n) \geq n$. Show that this is in fact an equality, by finding a function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ so that $\int_{S^n} f = 0$ and:

$$n \int_{S^n} f^2 = \int_{S^n} |\nabla_{S^n} f|^2.$$

Here ∇_{S^n} is the (Riemannian) spherical gradient, i.e. :

$$\nabla_{S^n} f(\theta) = Proj_{\theta^\perp} \nabla_{\mathbb{R}^{n+1}} f(\theta) \text{ for } \theta \in S^n \subset \mathbb{R}^{n+1},$$

where $\nabla_{\mathbb{R}^{n+1}} f$ denotes the usual gradient of f in \mathbb{R}^{n+1} , and $Proj_{\theta^\perp}$ denotes orthogonal projection onto $T_\theta S^n$, i.e. the n -dimensional subspace orthogonal to $\theta \in S^n \subset \mathbb{R}^{n+1}$.

(42) Use the generalized Bochner formula:

$$\frac{1}{2}\Delta_{g,\mu}|\nabla f|^2 = \langle \nabla f, \nabla \Delta_{g,\mu} f \rangle + \|Hess_g f\|_{HS}^2 + Ric_{g,\mu}(\nabla f, \nabla f),$$

to show the generalized Lichnerowicz estimate: On a complete weighted manifold (M^n, g, μ) , if $Ric_{g,\mu} \geq \lambda g > 0$, then $\lambda_{SG}(M, g, \mu) \geq \lambda$.

Guidance: you may assume (the correct fact) that the weighted Laplacian $-\Delta_{g,\mu}$ has discrete spectrum and in particular an eigenfunction corresponding to the first non-trivial eigenvalue λ_{SG} .

Deduce that for Gaussian space, $\lambda_{SG}(\mathbb{R}^n, |\cdot|, \gamma_n) \geq 1$. Show that in fact $\lambda_{SG}(\mathbb{R}^n, |\cdot|, \gamma_n) = 1$, by finding an appropriate test function f so that $Var_{\gamma_n}(f) = \int |\nabla f|^2 d\gamma_n$.

(43) Recall that on a weighted manifold (M, g, μ) , the Γ -operators are constructed as follows:

$$\Gamma_{i+1}(f, h) = \frac{1}{2}(L\Gamma_i(f, h) - \Gamma_i(Lf, h) - \Gamma_i(f, Lh))$$

with $\Gamma_0(f, h) = f \cdot h$. In particular, $\Gamma_1(f, h) = g(\nabla f, \nabla h)$ and $\Gamma_2(f, h) = tr_g(Hess_g f \cdot Hess_g h) + Ric_{g,\mu}(\nabla f, \nabla h)$. Recall that we set $\Gamma_i(f) := \Gamma_i(f, f)$, and write that $\Gamma_2 \geq \lambda \Gamma_1$ if this holds whenever evaluating any function f .

(a) Show that the assumption $\Gamma_2 \geq \lambda \Gamma_1$ for $\lambda \in \mathbb{R}$ implies the following two pointwise inequalities:

$$(2) \quad \frac{\exp(2\lambda t) - 1}{\lambda} |\nabla P_t f|^2 \leq P_t(f^2) - P_t(f)^2 \leq \frac{1 - \exp(-2\lambda t)}{\lambda} P_t(|\nabla f|^2).$$

(with the interpretation when $\lambda = 0$ in the limiting sense, e.g. $\frac{\exp(2\lambda t) - 1}{\lambda} \Big|_{\lambda=0} = 2t$). Guidance: consider the function $P_s((P_{t-s}f)^2)$, and repeat the proof we saw in class for showing the inequality between left and right expressions:

$$(3) \quad |\nabla P_t f|^2 \leq \exp(-2\lambda t) P_t(|\nabla f|^2).$$

(b) Deduce from part (a) an alternative proof of the generalized Lichnerowicz estimate of exercise 42.

(c) Show that any of the three inequalities appearing in (2) and (3) above are in fact equivalent to the property that $\Gamma_2 \geq \lambda \Gamma_1$.