CONVEX BODIES IN HIGH DIMENSION

(note that the statement of one exercise may be used for the proof of another).

(1) (a) Show that there is a one-to-one correspondence between origin-symmetric convex bodies K in $\mathbb{R}^n$ (i.e. compact convex subsets with non-empty interior), and normed spaces $(\mathbb{R}^n, \|\cdot\|_K)$, given by $\|x\|_K := \inf\{t > 0; x \in tK\}$.
   (b) Show that $K \subset L$ iff $\|x\|_K \geq \|x\|_L$ for all $x \in \mathbb{R}^n$.

(2) Recall that he Banach–Mazur distance between two convex bodies $K, L \subset \mathbb{R}^n$ was defined as:
$$d_{BM}(K, L) := \inf\left\{ab \ ; \ \frac{1}{a}L \subset T(K) \subset bL \ , \ T \in \text{Affine}(\mathbb{R}^n)\right\} \ ,$$
Where Affine($\mathbb{R}^n$) denotes all affine transformations of $\mathbb{R}^n$. Show that when $K = -K$ and $L = -L$, it is enough to just consider linear maps $T \in GL(n)$ above.

(3) Let Δ_n denote the regular simplex in $\mathbb{R}^n$. Show that $d_G(\Delta_n, D_n) = n$, where recall, the geometric distance is defined as:
$$d_G(K, L) := \inf\left\{ab \ ; \ \frac{1}{a}L \subset K \subset bL\right\} \ .$$
Hint: use the natural embedding of $\Delta_n \subset \mathbb{R}^{n+1}$ as $\Delta_n := conv(e_1, \ldots, e_{n+1})$.

(4) Recall that B_∞^n and B_1^n denote the unit-balls of ℓ_∞^n and ℓ_1^n, respectively.
   (a) Verify that $d_G(B_1^n, D_n) = \sqrt{n}$.
   (b) Verify that $d_G(B_\infty^n, D_n) = \sqrt{n}$.

(5) Let H denote the hyperplane perpendicular to the diagonal in $\mathbb{R}^n$, i.e. $H := (1/\sqrt{n}, \ldots, 1/\sqrt{n})^\perp$. Using the (local) Central-Limit-Theorem, calculate:
$$\lim_{n\to\infty} Vol([-1/2, 1/2]^n \cap H) \ .$$
Verify that indeed this value is $\leq \sqrt{2}$. (Here local CLT means that if $\{X_i\}$ are i.i.d. bounded random-variables with $E(X_1) = 0$ and $Var(X_1) = 1$, then the density at $t \in \mathbb{R}$ of $\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$ converges to the density of a standard Gaussian random-variable at t).

(6) We have proved the Brunn-Minkowski inequality for two n-dimensional rectangles with aligned axes. Use this as the basis of an induction, and prove the BM inequality for bodies $K = \cup_{i=1}^N A_i$ and $L = \cup_{j=1}^M B_j$, where A_i and B_j are n-dimensional rectangles (with axes aligned with the principle axes of $\mathbb{R}^n$) having mutually disjoint interiors.
Hint: use induction on $N + M$. Use the fact that if $N \geq 2$, one can always find a hyperplane H so that each of the collections $\{A_i \cap H^+\}$ and

$\{A_i \cap H^-\}$ are composed of $\leq N-1$ non-degenerate rectangles. Finally, translate the set L into a favorable position, so that $L \cap H^+$ and $L \cap H^-$ have convenient volumes.

(7) Prove that $det^{1/n}$ is concave on the class of $n \times n$ positive semi-definite matrices:
$$A, B \geq 0 \quad \Rightarrow det(A+B)^{1/n} \geq det(A)^{1/n} + det(B)^{1/n} \ .$$

(8) We've seen in class that the 1-D Brunn–Minkowski (BM) inequality implies the 1-D Prékopa–Leindler (PL) inequality. Generalize this implication to arbitrary dimension n using two different methods:
   (a) Method 1. Apply the BM inequality to the level-sets of h, f, g, and conclude using the 1-D PL inequality.

   (b) Method 2. First prove using the BM inequality that if:
   $$h(\lambda x + (1-\lambda)y)^{\frac{1}{k}} \geq \lambda f(x)^{\frac{1}{k}} + (1-\lambda) g(y)^{\frac{1}{k}} \quad \forall x, y \in \mathbb{R}^n$$
   for some natural number k, then:
   $$\left(\int_{\mathbb{R}^n} h\right)^{\frac{1}{n+k}} \geq \lambda \left(\int_{\mathbb{R}^n} f\right)^{\frac{1}{n+k}} + (1-\lambda) \left(\int_{\mathbb{R}^n} g\right)^{\frac{1}{n+k}} \ .$$
   (this is a particular case of the Borell / Brascamp-Lieb inequalities). Conclude the PL inequality by taking $k \to \infty$ and using that for $a, b > 0$:
   $$\lim_{\varepsilon \to 0} \left(\lambda a^{\varepsilon} + (1-\lambda) b^{\varepsilon}\right)^{1/\varepsilon} \to a^{\lambda} b^{1-\lambda} \ .$$

(9) We've seen in class that if $f : \mathbb{R}^n \to \mathbb{R}_+$ is a log-concave function:
$$\forall \lambda \in [0,1] \ \ \forall x, y \in \mathbb{R}^n \quad f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda} \ ,$$
then $\mu = f(x)dx$ is a log-concave measure:
$$\forall A, B \subset \mathbb{R}^n \quad \mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda} \mu(B)^{1-\lambda} \ .$$
Show the converse.

(10) (a) Show that if f is log-concave and integrable, then so are all of its marginals. In other words, show that if $f : \mathbb{R}^{n+m} \to \mathbb{R}_+$ is log-concave and integrable, then so is $h : \mathbb{R}^n \to \mathbb{R}_+$, where $h(x) := \int_{\mathbb{R}^m} f(x,y)dy$.
   (b) Deduce from part (a) that if $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are log-concave and integrable, then so is their convolution $f * g(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy$.

(11) Let $f : \mathbb{R} \to \mathbb{R}_+$. Show that if $0 < p < q$, then then the property "$f^q(t)$ is concave on its support" implies "$f^p(t)$ is concave on its support". Passing to the limit $p \to 0$, show that this implies that $\log f(t) : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave (i.e. that f is log-concave).

(12) Show that if A is compact in $\mathbb{R}^n$ then so is its Steiner symmetrization $S_H A$.

(13) Let A be a compact subset of $\mathbb{R}^n$. Show that $S_H A = A$ for all centered hyperplanes H, if and only if A is a centered Euclidean ball.

(14) Let $K \subset \mathbb{R}^n$ be a convex body, and let E be an m-dimensional subspace. Show that the function:

$$E \ni y \mapsto Vol(K \cap (y + E^{\perp}))^{\frac{1}{n-m}} ,$$

is concave on its support.

(15) Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a function so that $f^{1/k}$ is concave on its support, for some integer k. Denoting:

$$g(y) := \int_{y+E^{\perp}} f(x)dx ,$$

show that the function:

$$E \ni y \mapsto g(y)^{\frac{1}{n+k-m}} ,$$

is concave on its support. Taking $k \to \infty$, deduce that when f is log-concave, then so is its marginal g. Compare with Exercise 8(b) and with the previous exercise.

(16) Show that the isoperimetric inequality on $\mathbb{R}^n$:

$$|\partial A| \geq n\,|D_n|^{1/n}\,|A|^{(n-1)/n} ,$$

implies the Brunn-Minkowski inequality when one of the sets is a Euclidean ball D:

$$\mathrm{Vol}(A + D)^{1/n} \geq \mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(D)^{1/n} ,$$

(say for sets A with smooth boundary).

(17) Let K be an origin-symmetric convex body in $\mathbb{R}^n$ and E a linear subspace of dimension k. We proved in class the Rogers–Shephard inequality:

$$|K| \leq \left|K \cap E^{\perp}\right| |Proj_E K| \leq \binom{n}{k} |K| .$$

(a) Provide examples demonstrating the sharpness of each inequality.
(b) As a corollary or RS, deduce that if K is not-necessarily origin-symmetric, then:

$$|K + (-K)| \leq \binom{2n}{n} |K| .$$

Hint: construct an appropriate $L \subset \mathbb{R}^{2n}$ and use RS with an appropriate E. Note that for the RHS of the RS inequality, no assumption on origin-symmetric is required.

(18) Prove that $\mathbb{R}_+ \ni r \mapsto Vol(K \cap rD_n)^{1/n}$ is a concave function, for any convex K in $\mathbb{R}^n$.

(19) Recall the correspondence between origin-symmetric convex bodies K and norms on $\mathbb{R}^n$, described in Exercise 1. Show that:
(a) $\|x\|_{K\cap L} = \max(\|x\|_K , \|y\|_L)$.

(b) $\|z\|_{conv(K \cup L)} = \inf \{\|x\|_K + \|y\|_L \,;\, z = x + y\}$.
(c) $\|z\|_{K+L} = \inf \{\max(\|x\|_K , \|y\|_L); z = x + y\}$.

(20) Recall the definition of the dual norm: $\|y\|^* := \sup_{\|x\| \leq 1} |\langle x, y \rangle|$. Show that:
  (a) $\|\cdot\|^*$ is indeed a norm.
  (b) $(\|\cdot\|^*)^* = \|\cdot\|$. Hint: prove that if $x \notin K$ then there exists a separating hyperplane.
  (c) $\|\cdot\|_1 \leq \|\cdot\|_2$ implies $\|\cdot\|_1^* \geq \|\cdot\|_2^*$.

(21) Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and recall that the unit-ball of $\|\cdot\|_K^*$ is polar-set K°. Show that:
  (a) $(K_1 \cap K_2)^\circ = conv(K_1^\circ \cup K_2^\circ)$.
  (b) $h_{K_1+K_2} = h_{K_1} + h_{K_2}$, where $h_L = \|\cdot\|_L^*$ is the support function of L.
  (c) If $K = [-a\theta, a\theta]$, $\theta \in S^{n-1}$, then $K^\circ = \{x \in \mathbb{R}^n \; ; \; |\langle x, \theta \rangle| \leq 1/a\}$.
  (d) $K \subset L$ implies $L^\circ \subset K^\circ$.
  (e) If $T \in GL(n)$ then $(TK)^\circ = T^{-*}(K^\circ)$, where $T^{-*} = (T^{-1})^*$. In particular, $(\lambda K)^\circ = \lambda^{-1} K^\circ$.
  (f) Conclude that $d_{BM}(K, L) = d_{BM}(K^\circ, L^\circ)$.

(22) Show that $M^*(S_H K) \leq M^*(K)$, where $S_H K$ denotes the Steiner symmetral of K with respect to an arbitrary linear hyperplane H. As a corollary, deduce Urysohn's inequality: $|K| = |D_n|$ implies $M^*(K) \geq M^*(D_n) = 1$.

(23) Let X be a G-homogeneous space, where G is a compact Hausdorff topological group. Recall that the Haar measure μ_X on X is defined by: $\int_X f(x) d\mu_X(x) = \int_G f(gx_0) d\mu_G(g)$, for any fixed $x_0 \in X$. Show that μ_X is the unique (up to positive multiple) G-left-invariant measure on X.

(24) (a) Let $\theta_0 \in S^{n-1}$, and let $G_{n,k}$ denote the Grassmanian of all k-dimensional linear subspaces of $\mathbb{R}^n$. Show that the space $G_{n,k}^{\theta_0} := \{E \in G_{n,k} \; ; \; \theta_0 \in E\}$ is O_{θ_0}-homogeneous for an appropriate compact group O_{θ_0}.
  (b) Using uniqueness of the Haar measure, show that for any continuous function f on $G_{n,k}$:
$$\int_{S^{n-1}} \int_{G_{n,k}^\theta} f(E) d\mu_{G_{n,k}^\theta}(E) d\mu_{S^{n-1}}(\theta) = \int_{G_{n,k}} f(E) d\mu_{G_{n,k}}(E) \; ,$$
  where μ_X denotes the Haar **probability** measure on the compact space X.
  (c) Similarly, show that for any continuous function f on S^{n-1}:
$$\int_{G_{n,k}} \int_{S(E^k)} f(\theta) d\mu_{S(E^k)}(\theta) d\mu_{G_{n,k}}(E^k) = \int_{S^{n-1}} f(\theta) d\mu_{S^{n-1}}(\theta) \; ,$$
  where $E^k \in G_{n,k}$ and $S(E^k) := E^k \cap S^{n-1}$.

(25) Recall that Steiner's formula states that:
$$Vol(K + tD_n) = \sum_{i=0}^{n} \binom{n}{i} W_{n-i}(K) t^i \; .$$

Let K be a smooth convex body. Prove using Steiner's formula and integration in polar coordinates that $W_1(K) = Vol(D_n)M^*(K)$, where recall:

$$M^*(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) \ .$$

Here σ is the Haar probability measure on S^{n-1} and h_K is the support function of K.

(26) Using $Vol((K + tD_n) + rD_n) = Vol(K + (t + r)D_n)$ and Steiner's formula, deduce that:

$$W_j(K + rD_n) = \sum_{i=0}^{j} \binom{j}{i} W_{j-i}(K) r^i \ .$$

(27) Recall that the Alexandrov–Fenchel inequality states that:

$$V(K_1, K_2, L_3, \ldots, L_n)^2 \geq V(K_1, K_1, L_3, \ldots, L_n)V(K_2, K_2, L_3, \ldots, L_n) \ ,$$

where V denotes mixed-volume. We use the notation $V(K; i, L; n - i)$ to denote the mixed volume of the tuple where K appears i times and L appears $n - i$ times. Using only this, deduce:

(a) $V(K; i, L; n - i) \geq Vol(K)^{i/n} Vol(L)^{(n-i)/n}$.

(b) More generally, deduce Minkowski's inequality:

$$V(K_1, \ldots, K_n) \geq \left(\Pi_{i=1}^{n} Vol(K_i)\right)^{1/n} \ .$$

(c) Specializing (a) to the case $i = n - 1$, we have Minkowski's inequality:

$$V(K; n - 1, L; 1) \geq Vol(K)^{(n-1)/n} Vol(L)^{1/n} \ .$$

Note and explain why the case of $L = D_n$ corresponds to the Euclidean isoperimetric inequality (for convex bodies).

(d) Use the inequality in (c) above and the multi-linearity of the mixed-volumes V to deduce the Brunn–Minkowski inequality:

$$Vol(K + L)^{1/n} \geq Vol(K)^{1/n} + Vol(L)^{1/n} \ .$$

(e) Generalize all of the above to show that:

$$W_i(K + L)^{1/i} \geq W_i(K)^{1/i} + W_i(L)^{1/i} \ , \ \forall i = 1, \ldots, n \ ,$$

where recall $W_i(K) = V(K; i, D_n; n - i)$.

(28) Calculate the isotropic constant of the Euclidean ball D_n and the cross-polytope B_1^n having volume 1 in $\mathbb{R}^n$. Find their asymptotic values as $n \to \infty$.

(29) Complete the proof of John's theorem, stating that if $D_n \subset K$ is the maximal volume ellipsoid inside an origin-symmetric convex body K in $\mathbb{R}^n$, then $K \subset \sqrt{n}D_n$. Do this by directly arguing that otherwise, there would exist a point $x_0 \notin K$ with $|x_0| > \sqrt{n}$, and hence by convexity $K \supset C := conv(D_n \cup \{\pm x_0\})$; a contradiction will follow if an ellipsoid is found inside C having even larger volume than D_n.

(30) (a) Show that $K = \sqrt{n}B_1^n$ is in John's position (can use the characterization we learned in class via the isotropic measure supported on contact points of ∂D_n and ∂K).
(b) Deduce by duality that the minimal-volume ellipsoid containing B_∞^n is $\sqrt{n}D_n$.
(c) Deduce from (a) and (b) that $d_{BM}(B_\infty^n, D_n) = \sqrt{n}$.

(31) Show that when k divides n, the estimate of K. Ball $Vol([-1/2,1/2]^n \cap E) \leq (n/k)^{k/2}$ for a subspace E of dimension k is sharp.

(32) Derive using the Brascamp-Lieb Theorem the sharp constant in Young's inequality on $\mathbb{R}$:

$$\|f * g\|_{L^s} \leq C_{p,q,s} \|f\|_{L^p} \|g\|_{L^q} \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s} \ , \ p,q,s \geq 1 \ .$$

(33) (a) Calculate the volume of B_p^n, the unit-ball of ℓ_p^n, by integrating the measure $\exp(-\|x\|_{\ell_p^n}^p)dx$ on $\mathbb{R}^n$.
(b) Deduce that when $p \in [1,2]$, the volume-ratio of B_p^n is bounded by a numeric constant, independent of dimension.

(34) Consider the metric-measure space (S^n, d, μ_{S^n}), where $\mu = \mu_{S^n}$ it the corresponding Haar probability measure, and d is the geodesic distance on S^n. Let $f : S^n \to \mathbb{R}$ be a 1-Lipschitz function, let $m(f)$ denote its median, and define $E(f) = \int f d\mu$.
(a) Show that $|E(f) - m(f)| \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
(b) Show that $0 \leq \sqrt{E(f^2)} - E(|f|) \leq \frac{C}{\sqrt{n}}$ for some constant $C > 0$.
(c) Deduce concentration around $E(f)$, and if $f \geq 0$, also around $\sqrt{E(f^2)}$. In other words, show that:

$$\mu(x \in S^n \ ; \ |f(x) - A_f| \geq r) \leq C \exp(-\frac{n-1}{2} r^2) \ ,$$

where A_f is either $E(f)$, and when $f \geq 0$, also $\sqrt{E(f^2)}$, for some constant $C > 0$.
(d) Show that if f is L-Lipschitz, then:

$$\mu(x \in S^n \ ; \ f(x) - m(f) \geq r) \leq \sqrt{\pi} 2 \exp(-\frac{n-1}{2L^2} r^2) \ .$$

(e) More generally, show that:

$$\mu(x \in S^n \ ; \ f(x) - m(f) \geq \omega_f(r)) \leq \sqrt{\pi} 2 \exp(-\frac{n-1}{2} r^2) \ ,$$

where $\omega_f(r) = \sup\{|f(x) - f(y)| \ ; \ d(x,y) \leq r\}$ denotes the modulus of continuity of f.

(35) Show that $\mu_{S^n}(B_\theta(x_0)) \geq c\sqrt{n-1}\ \theta \sin^{n-1}(\theta)$, where $B_\theta(x_0)$ is a geodesic ball of radius θ on S^n, $x_0 \in S^n$ is any point, and $c > 0$ is a constant.

(36) Show that the Gaussian isoperimetric inequality on $(\mathbb{R}^n, |\cdot|, \gamma_n)$, where γ_n denotes the standard Gaussian measure on $\mathbb{R}^n$, namely:

$$\gamma_n(A) = \gamma_n(H) \quad \Rightarrow \quad \gamma_n^+(A) \geq \gamma_n^+(H) \ ,$$

implies:

$$\gamma_n(A) = \gamma_n(H) \quad \Rightarrow \quad \gamma_n(A_r) \geq \gamma_n(H_r) \ , \ \forall r > 0 \ .$$

Here H is (any) half-plane, $\gamma_n^+(C)$ denote the Gaussian boundary measure of a Borel set $C \subset \mathbb{R}^n$, and C_r denote the r-extension of C.
Guidance: mimic the proof we saw in class for the Sphere.

(37) Let A denote a $k \times n$ random matrix with i.i.d. standard Gaussian entries. Show that with very high-probability (quantify this!), the map $T : \mathbb{R}^n \to \mathbb{R}^k$ given by $T(x) := \frac{1}{\sqrt{k}} Ax$ is a good "Johnson-Lindenstrauss" map, i.e.:

$$(1-\varepsilon)|z|_{\mathbb{R}^n} \leq |T(z)|_{\mathbb{R}^k} \leq (1+\varepsilon)|z|_{\mathbb{R}^n}$$

with very high-probability for a fixed $z \in \mathbb{R}^n$.

(38) Assume that on a metric-measure (probability) space (Ω, d, μ) we know that:

$$\forall A \subset \Omega \quad \mu(A) \geq 1/2 \quad \Rightarrow \quad \mu(\Omega \setminus A_r) < K(r) \ \ \forall r > 0 \ .$$

Show that:

$$\forall A \subset \Omega \quad \mu(A) \geq K(\varepsilon_0) \quad \Rightarrow \quad \mu(\Omega \setminus A_{r+\varepsilon_0}) < K(r) \ \ \forall r > 0 \ .$$

Deduce that on S^n, we have the following concentration property:

$$\mu(A) \geq \sqrt{\pi/8}\exp(-\frac{n-1}{2}\varepsilon_0^2) \quad \Rightarrow \quad \mu(S^n \setminus A_{2\varepsilon_0}) \leq \sqrt{\pi/8}\exp(-\frac{n-1}{2}\varepsilon_0^2) \ .$$

(39) Let $k = k(n-1, \varepsilon)$ as defined in class, namely a value of k which guarantees that for any continuous function on S^{n-1}:

$$\mu_{G_{n,k}}\left\{E^k \in G_{n,k} \ ; \ \forall x \in S(E^k) \ \ |f(x) - m(f)| \leq \omega_f(\varepsilon)\right\} \geq 1 - \exp(-\frac{n-2}{4}\varepsilon^2) \ .$$

Assume that $A \subset S^{n-1}$ is such that for every $E^k \in G_{n,k}$, $A \cap E^k \neq \emptyset$. Then there exists a $E_0^k \in G_{n,k}$ so that $E_0^k \cap S^{n-1} \subset A_{2\varepsilon}$.
Guidance: define the right Lipschitz function f and use a similar argument to the one in the previous exercise.

(40) Recall that:

$$M_p(K) := \left(\int_{S^{n-1}} \|x\|_K^p \, d\mu_{S^{n-1}}(x)\right)^{1/p} \ , \ \Gamma_p(K) := \left(\int_{\mathbb{R}^n} \|x\|_K^p \, d\gamma_n(x)\right)^{1/p} \ ,$$

and that we showed in class that $\Gamma_2(K) = \sqrt{n} M_2(K)$ for any (say origin-symmetric) convex K.

(a) Show by direct calculation that $\Gamma_1(K) \simeq \sqrt{(n)} M_1(K)$. Here $A \simeq B$ means that the ratio of A and B is bounded from above and from below by two numerical constants, independent of the dimension.

(b) Show the "reverse Jensen inequality": $M_p(K) \leq C\sqrt{p}M_1(K)$.
Guidance: we showed that:

$$\mu_{S^{n-1}}(|\|x\|_K - M_1(K)| \geq tM_1(K)) \leq C\exp(-cnt^2(M_1(K)/b(K))^2) \leq C\exp(-c't^2)\ ;$$

now repeat the proof of the reverse Holder inequality we gave in class for semi-norms on log-concave distributions (corollary of Borell's Lemma).

(c) Deduce (a) again without performing any calculations.

(41) Prove the following estimates:

$$\frac{1}{\sqrt{2\pi}}\frac{t}{t^2+1}\exp(-t^2/2) \leq \mathrm{Prob}(\gamma_1 > t) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{t}\exp(-t^2/2) \quad \forall t > 0\ .$$

Hint for the left-hand inequality - try to take derivatives.

(42) Show that for any $E \in G_{n,k}$, $\Gamma_2(K) \geq \Gamma_2(K \cap E)$, and hence $M_2(K \cap E) \leq \sqrt{n/k}M_2(K)$. Using $k = 1$, deduce that $b(K) \leq \sqrt{n}M_2(K)$.
Guidance: repeat the proof of the proposition where we showed that in John's position, $M(K)/b(K) \geq c\sqrt{\log n}/\sqrt{n}$.

(43) Show that $\mathrm{Med}(\|\cdot\|)\mathrm{Med}(\|\cdot\|^*) \geq 1$. Here $\mathrm{Med} = m$ denotes a median of its argument on S^{n-1}.

(44) Show that for any convex set K and Euclidean ball D, $N(K, D) = \bar{N}(K, D)$, where N is the covering number, and $\bar{N}$ is the covering number where the points are required to all lie inside K.

(45) Show that if K, T, D are three origin-symmetric convex bodies, then for any $z \in \mathbb{R}^n$:

$$|(K \cap (z + D)) + T| \leq |(K \cap D) + T|\ .$$

Hint: use Brunn's concavity principle and the origin-symmetry.

(46) Show that the following two statements are equivalent for an origin-symmetric convex body K in $\mathbb{R}^n$ with $|K| = |D_n|$.
(a) There exists a constant $C_1 > 0$ so that $N(K, D_n) \leq \exp(C_1 n)$.
(b) There exists a constant $C_2 > 0$ so that $|K \cap D_n| \geq \exp(-C_2 n)\,|D_n|$.
The equivalence is in the sense that C_1 depends solely on C_2, and vice-versa.
Hint: use the covering estimates we learned in class.

(47) Recall that the volume-ratio of a convex body T is defined as:

$$v.r.(T) := \inf_{\mathbb{E} \subset T} \left(|T| \,/\, |\mathcal{E}|\right)^{1/n}\ ,$$

where the infimum runs over all ellipsoids $\mathcal{E}$ contained in T.

Assume that K satisfies $|K| = |D_n|$ and $N(K, D_n) \leq \exp(C_1 n)$. Setting $T := conv(K^\circ \cup D_n)$, show that $v.r.(T) \leq C_3$, where $C_3 > 1$ is some fixed constant.

Hint: use the previous exercise and the Blaschke–Santalo inequality for origin-symmetric convex bodies $|T|\,|T^\circ| \leq |D_n|^2$.

(48) Assume that K satisfies $|K| = |D_n|$ and $N(K, D_n) \leq \exp(C_1 n)$. Show that $N(K^\circ, D_n) \leq \exp(C_4 n)$.
Hint: use the previous exercise.